# Global strong solution with BV derivatives to singular solid-on-solid model with exponential nonlinearity 

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#### Abstract

In this work, we consider the one dimensional very singular fourth-order equation for solid-on-solid model in attachment-detachment-limit regime with exponential nonlinearity $$
h_{t}=\nabla \cdot\left(\frac{1}{|\nabla h|} \nabla e^{\frac{\delta E}{\delta h}}\right)=\nabla \cdot\left(\frac{1}{|\nabla h|} \nabla e^{-\nabla \cdot\left(\frac{\nabla h}{|\nabla h|}\right)}\right)
$$ where total energy $E=\int|\nabla h|$ is the total variation of $h$. Using a logarithmic correction for total energy $E=\int|\nabla h| \ln |\nabla h| \mathrm{d} x$ and gradient flow structure with a suitable defined functional, we prove the one dimensional evolution variational inequality solution preserves a positive gradient $h_{x}$ which has upper and lower bounds but in BV space. We also obtain the global strong solution to the solid-on-solid model which allows an asymmetric singularity $h_{x x}^{+}$to happen.


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## 1. Introduction

### 1.1. Background

Epitaxial growth on crystal surface is an important nanoscale phenomena which has attracted lots of attention due to its application in industry and in manufacture of some typical experimental materials. We refer to [17,27] for more physical description.

In this paper, we focus on dynamic process for solid on solid (SOS) model on crystal surface, where adatoms detach from above, diffuse on the substrate and then are absorbed at another position. There are some researches on the SOS model from microscopic viewpoint and derivation of continuum limit from mesoscopic level; see [4,7,18,24,28]. The kinetic process can also be described using macroscopic variable, height profile $h(x, t)$ of a solid film. Here we directly write down the evolution equation for surface height $h(x, t)$ using conservation law of mass

$$
h_{t}+\nabla \cdot J=0,
$$

where

$$
J=-M(\nabla h) \nabla \rho_{s}
$$

is the adatom flux by Fick's law [24], the mobility function $M(\nabla h)$ is a functional of $\nabla h$ and $\rho_{s}$ is the local equilibrium density of adatoms. By the Gibbs-Thomson relation [19,25,24], which is connected to the theory of molecular capillarity, the corresponding local equilibrium density of adatoms is given by

$$
\rho_{s}=\rho^{0} e^{\frac{\mu}{k T}},
$$

where $\rho^{0}$ is a constant reference density, $T$ is the temperature, $k$ is the Bolzmann constant and $\mu$ is the chemical potential.

Now we consider the expression of the chemical potential $\mu$, the rate of change in the surface energy per atom. For a physical constant $L$, we impose one dimensional screw periodic boundary condition for simplicity, i.e.

$$
\begin{equation*}
h(x+L)=h(x)+1 \quad \text { for a.e. } x \in \mathbb{R}, \tag{1.1}
\end{equation*}
$$

which means $h_{x}$ is $L$-periodic. Denote the domain for one period as $\mathbb{T}:=[0, L)$. The general total energy for epitaxial growth is

$$
\begin{equation*}
E=\frac{1}{p} \int_{\mathbb{T}}|\nabla h|^{p} \mathrm{~d} x \tag{1.2}
\end{equation*}
$$

for some $p \geq 1$ and the corresponding chemical potential is

$$
\begin{equation*}
\mu=\frac{\delta E}{\delta h}=-\nabla \cdot\left(|\nabla h|^{p-2} \nabla h\right) . \tag{1.3}
\end{equation*}
$$

Hence the general evolution equation becomes

$$
\begin{equation*}
h_{t}=\nabla \cdot\left(M(\nabla h) \nabla e^{\frac{\mu}{K T}}\right)=\nabla \cdot\left(M(\nabla h) \nabla e^{-\nabla \cdot\left(|\nabla h|^{p-2} \nabla h\right)}\right), \tag{1.4}
\end{equation*}
$$

where the mobility $M(\nabla h)$ is a constant in diffusion-limit (DL) regime, while the mobility $M(\nabla h)=\frac{1}{|\nabla h|}$ in the attachment-detachment-limit (ADL) regime; see $[7,9,18,20,24,26]$ and the references in there.

Difficulties and references in the continuum framework. In the previous researches, the exponential form of chemical potential $e^{\mu / k T}$ is regarded as linear in chemical potential $e^{\mu / k T} \approx$ $1+\mu / k T$ under the hypothesis $|\mu| \ll k T$. When $p>1$, we refer to [1,6,9,10,23] for analytical results including existence, uniqueness and long time behaviors in DL regime and ADL regime. General speaking, the ADL model is harder than DL model due to the singular mobility $\frac{1}{|\nabla h|}$ so the global monotone solution is understood in almost everywhere sense in [9,10]. For the case $p=1$, the total energy and chemical potential become the total variation of $h$ (see physical derivation from microscopic viewpoint by bond counting in [22]), i.e.

$$
\begin{equation*}
E=\int_{\mathbb{T}}|\nabla h|, \quad \mu=\frac{\delta E}{\delta h}=-\nabla \cdot\left(\frac{\nabla h}{|\nabla h|}\right) \tag{1.5}
\end{equation*}
$$

After linearization, this kind of fourth-order singular equation in DL regime is regarded as $H^{-1}$ gradient flow for the BV seminorm $\int|\nabla h|$. The discontinuous solution is studied in [12] and the flattening effect in finite time is proved in [13]; see also [14,15] for further development in $H^{-s}$ space and other boundary conditions. However, the method therein works only for DL regime whose mobility is a constant and the evolution in ADL regime is still an open question for $p=1$. More recently, the original exponential equation (1.4) in DL regime is studied in [8,22,16] for $p=2$ and in [30] for $p \in(1,2]$, where the existence of strong solution with latent singularity and global solution starting from small data are established. For $p=1$ in DL regime, [21] constructs a explicit solution to demonstrate the asymmetry of height profile due to the exponential effect. No matter with or without linearization, those methods in DL regime for $p=1$ more or less rely on the total variation flow structure of the PDE so it fails to work for ADL regime. To our best knowledge, there is no result for the evolution equation in ADL regime with $p=1$

$$
\begin{equation*}
h_{t}=\nabla \cdot\left(M(h) \nabla e^{\frac{\mu}{K T}}\right)=\nabla \cdot\left(\frac{1}{|\nabla h|} \nabla e^{-\nabla \cdot\left(\frac{\nabla h}{|\nabla h|}\right)}\right) \tag{1.6}
\end{equation*}
$$

which is a very singular fourth order equation with exponential nonlinearity.
Logarithmic correction and explanation from mesoscopic view. From the mesoscopic view we can regard the surface evolution equation as continuum limit of discrete Burton-CabreraFrank (BCF) model [3,7,9], which tracks the dynamics of positions of each step $x_{i}$ with height $h_{i}=h_{\text {ref }}+\frac{i}{N}$. Here $N$ is the number of steps in one period and will go to $+\infty$ in the continuum limit. In ADL regime, the dynamics of $x_{i}$ can be expressed by

$$
\begin{equation*}
\frac{\mathrm{d} x_{i}}{\mathrm{~d} t}=N\left[\left(f_{i+1}-f_{i}\right)-\left(f_{i}-f_{i-1}\right)\right], \quad i=1, \cdots, N \tag{1.7}
\end{equation*}
$$

with only repulsive interaction between the nearest step

$$
f_{i}:=-\left(\frac{1}{x_{i+1}-x_{i}}-\frac{1}{x_{i}-x_{i-1}}\right)=\frac{\partial E_{i}}{\partial x_{i}},
$$

which is actually the dominated elastic interaction [29] in BCF step model depending on the distance between steps. Then the corresponding discrete energy is

$$
E_{i}=\frac{1}{2} \sum_{i=1}^{N} \sum_{|j-i|=1} \ln \left|x_{i}-x_{j}\right|
$$

The corresponding continuum interaction function $f$ in the limit PDE is $f=-\nabla(\ln |\nabla h|)$ (see detailed consistent check in [7]). This inspires us that we shall use a logarithmic factor to adjust the total energy for the case of $p=1$. Therefore, we take the total energy with logarithmic correction as

$$
\begin{equation*}
E(h):=\int|\nabla h| \ln |\nabla h| \mathrm{d} x, \quad \mu:=\frac{\delta E(h)}{\delta h}=-\nabla \cdot\left(\frac{\nabla h}{|\nabla h|}(\ln |\nabla h|+1)\right) . \tag{1.8}
\end{equation*}
$$

This kind of logarithmic correction is also used for the linearized surface evolution equation in [11] since the logarithmic correction is negligible for small surface gradients. The surface height equation turns out to be

$$
\begin{equation*}
h_{t}=\nabla \cdot\left(M(h) \nabla e^{\frac{\mu}{K T}}\right)=\nabla \cdot\left(\frac{1}{|\nabla h|} \nabla e^{-\nabla \cdot\left(\frac{\nabla h}{|\nabla h|}(\ln |\nabla h|+1)\right)}\right) . \tag{1.9}
\end{equation*}
$$

Results and methods. In this paper, we start with the simplest situation: one dimensional case with monotone initial data, i.e. $\partial_{x} h_{0}>0$. If we can prove $h_{x}>0$ for all the time, then we obtain a mathematical validation for surface height equation (1.9), i.e.

$$
\begin{equation*}
h_{t}=\nabla \cdot\left(M(h) \nabla e^{\mu}\right)=\left(\frac{1}{h_{x}}\left(e^{-\left(\ln h_{x}\right)_{x}}\right)_{x}\right)_{x} \tag{1.10}
\end{equation*}
$$

with $\mu=\frac{\delta E(h)}{\delta h}=-\left(\ln h_{x}\right)_{x}$. Specifically, we investigate the existence and uniqueness of the evolution variational inequality (EVI) solution and monotone strong solution to (1.9) with a monotone initial data; see Theorem 2.5 and Theorem 3.1 separately. We first observe the $L^{2}$ gradient flow structure by defining a proper, lower semi-continuous convex functional $\phi$. However due to the asymmetric effect brought by exponential nonlinearity [21], we shall allow a latent singularity for $h_{x x}$ and define the convex functional only on the absolutely continuous part of $h_{x x}$; see rigorous definition in (2.4). Then thanks to the detailed properties for the convex functional $\phi$ and the bound for $h_{x}$ provided by one dimensional BV space, we can apply the gradient flow method in metric space [2] to obtain the EVI solution $h$ whose gradient is in BV space and has upper/lower bound. To further explore the strong solution with latent singularity to (1.10) in the sense that the equation holds almost everywhere (see Definition 2), we carefully characterize the sub-differential of $\phi$ by first carry on the calculations in some dense set then prove the sub-differential $\partial \phi$ is single-valued; see Theorem 3.1. We call the singular part of $\left(\ln h_{x}\right)_{x}$
as latent singularity because the singularity does not destroy the evolution of the solution but it is not removable and leads to asymmetry behaviors for convex/concave parts of $h$ [21]. In the end, the notion of strong solution to this singular PDE (1.10) is understood in the sense that the equation holds almost everywhere after removing the singular part of $\left(\ln h_{x}\right)_{x}$. That is to say, the singular PDE can be understood as a limit of a regularized problem for all time although there is a singularity at finite time.

### 1.2. Gradient flow in $L^{2}(\mathbb{T})$

Let us first define formally a convex functional with some formal observations and recast (1.10) into a $L^{2}(\mathbb{T})$ gradient flow. Let $\phi$ be

$$
\begin{equation*}
\phi(h):=\int_{\mathbb{T}} e^{-\left(\ln h_{x}\right)_{x}} \mathrm{~d} x \tag{1.11}
\end{equation*}
$$

The variation of $\phi$ is

$$
\frac{\delta \phi}{\delta h}=-\left(\frac{1}{h_{x}}\left(e^{-\left(\ln h_{x}\right)_{x}}\right)_{x}\right)_{x}
$$

and then formally we have

$$
\begin{equation*}
h_{t}=-\frac{\delta \phi}{\delta h} . \tag{1.12}
\end{equation*}
$$

To study the monotone strong solution to (1.10), we plan to apply the gradient flow theory in metric space $L^{2}(\mathbb{T})$. We will define $\phi(h)$ rigorously later in (2.4). First, we need to clarify the working space associated with proper topology. Let us first see some inspiring observations.

Observation 1 (Conservation laws). Thanks to the screw periodic assumption (1.1), we have

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \int_{\mathbb{T}} h \mathrm{~d} x=0 \tag{1.13}
\end{equation*}
$$

which implies $\int_{\mathbb{T}} h \mathrm{~d} x=\int_{\mathbb{T}} h_{0} \mathrm{~d} x$. Moreover from

$$
\int_{\mathbb{T}} h_{x x} \mathrm{~d} x=0
$$

we know

$$
\begin{equation*}
\int_{\mathbb{T}}\left(h_{x x}\right)^{+} \mathrm{d} x=\int_{\mathbb{T}}\left(h_{x x}\right)^{-} \mathrm{d} x . \tag{1.14}
\end{equation*}
$$

Here $\left(h_{x x}\right)^{-}$is the negative part of $h_{x x}$ and $\left(h_{x x}\right)^{+}$is the positive part of $h_{x x}$. In fact, the notation of integration is just formal for now and we will see $\left(h_{x x}\right)^{+}$could be Radon measure later.

Observation 2 (Dissipation inequalities). From the gradient flow structure (1.12),

$$
\begin{equation*}
\frac{\mathrm{d} \phi}{\mathrm{~d} t}=\int_{\mathbb{T}} \frac{\delta \phi}{\delta h} h_{t} \mathrm{~d} x=-\int_{\mathbb{T}}\left|\frac{\delta \phi}{\delta h}\right|^{2} \mathrm{~d} x=-\int_{\mathbb{T}} h_{t}^{2} \mathrm{~d} x \leq 0 \tag{1.15}
\end{equation*}
$$

which gives the observation

$$
\phi(u(t)) \leq \phi(u(0)) \quad \text { for any } t \geq 0 .
$$

Therefore we obtain uniform estimate

$$
\begin{aligned}
\int_{\mathbb{T}}\left(\left(\ln h_{x}\right)_{x}\right)^{-} \mathrm{d} x & \leq \int_{\mathbb{T} \cap\left(\left(\ln h_{x}\right)_{x}\right)^{->}} e^{\left(\left(\ln h_{x}\right)_{x}\right)^{-}} \mathrm{d} x \leq \int_{\mathbb{T}} e^{\left(\left(\ln h_{x}\right)_{x}\right)^{--\left(\left(\ln h_{x}\right)_{x}\right)^{+}} \mathrm{d} x} \\
& =\int_{\mathbb{T}} e^{-\left(\ln h_{x}\right)_{x}} \mathrm{~d} x=\phi(h(t)) \leq \phi(h(0))
\end{aligned}
$$

where $\left(\left(\ln h_{x}\right)_{x}\right)^{-}$denotes the negative part of $\left(\ln h_{x}\right)_{x}$ and $\left(\left(\ln h_{x}\right)_{x}\right)^{+}$is the positive part of $\left(\ln h_{x}\right)_{x}$. Thanks to the screw periodic assumption (1.1), we have

$$
\begin{equation*}
\frac{1}{2}\left\|\left(\ln h_{x}\right)_{x}\right\|_{L^{1}(\mathbb{T})}=\left\|\left(\left(\ln h_{x}\right)_{x}\right)^{-}\right\|_{L^{1}(\mathbb{T})}=\left\|\left(\left(\ln h_{x}\right)_{x}\right)^{+}\right\|_{L^{1}(\mathbb{T})} \leq \phi(h(0)) \tag{1.16}
\end{equation*}
$$

However, since $L^{1}$ is non-reflexive Banach space, the uniform bound of $L^{1}$ norm does not prevent $\left(\ln h_{x}\right)_{x}$ being a Radon measure. This gives us the idea to carry on all the calculations in BV space, i.e. $\ln h_{x} \in B V(\mathbb{T})$; see explicit definitions in Section 2.

Outlines. The rest of this paper is organized as follows. We will define functional $\phi$ and establish the gradient flow structure rigorously in Section 2.1 and Section 2.2. Then after exploring some properties of $\phi$ in Section 2.3, we will prove the existence of EVI solution in Section 2.4. Section 3 is devoted to obtain the strong solution with latent singularity to (1.10).

## 2. Variational inequality solution

### 2.1. Preliminaries

We first introduce the spaces we will work in. Notice the invariant property of (1.10) if we add a constant $c$ to solution $h$ and (1.1). Therefore without loss of generality, we consider $h$ with mean value zero. Let

$$
\begin{equation*}
H:=\left\{u \in L^{2}(\mathbb{T}) ; \int_{\mathbb{T}} u \mathrm{~d} x=0\right\} \tag{2.1}
\end{equation*}
$$

endowed with the standard scalar product $\langle u, v\rangle_{H}:=\int_{\mathbb{T}} u v \mathrm{~d} x$. Here $u \in L^{2}(\mathbb{T})$ means $u \in$ $L^{2}(0, L)$ satisfies the screw periodic boundary condition (1.1).

As in the Observation 2, since $L^{1}$ is not reflexive Banach space and has no weak compactness, we work in a larger space, BV space. Denote $\mathcal{M}$ as the space of finite signed Radon measures and $\|\cdot\|_{\mathcal{M}(\mathbb{T})}$ as the total variation of the measure. Define Banach space

$$
\begin{equation*}
V:=\left\{u \in H ; u_{x} \in B V(\mathbb{T})\right\} \tag{2.2}
\end{equation*}
$$

Endow $V$ with the norm

$$
\|u\|_{V}:=\|u\|_{L^{2}(\mathbb{T})}+\left\|u_{x x}\right\|_{\mathcal{M}(\mathbb{T})}
$$

which is equivalent to the norm $\left\|u_{x x}\right\|_{\mathcal{M}(\mathbb{T})}$ due to Poincaré's inequality for mean value zero function.

Next, from Observation 2 we expect $\ln h_{x} \in B V(\mathbb{T}) \hookrightarrow L^{\infty}(\mathbb{T})$, which implies there will be a lower/upper bound for $h_{x}$. Therefore we expect there are constants $c_{1}, c_{2}$ such that $c_{1} \leq h_{x} \leq c_{2}$. Then the uniform estimate

$$
\left\|\left(\ln h_{x}\right)_{x}\right\|_{\mathcal{M}(\mathbb{T})}=\left\|\frac{h_{x x}}{h_{x}}\right\|_{\mathcal{M}(\mathbb{T})} \leq \phi(h(0))
$$

will lead to a uniform bound for $\left\|h_{x x}\right\|_{\mathcal{M}(\mathbb{T})}$. Since $h_{x x}$ can be a Radon measure, we need to make those formal observations rigorous in Section 1.2 by first defining $\phi$ properly. Notice for any $\mu \in \mathcal{M}$, from [5, p. 42], we have the decomposition

$$
\begin{equation*}
\mu=\mu_{\|}+\mu_{\perp} \tag{2.3}
\end{equation*}
$$

with respect to the Lebesgue measure, where $\mu_{\|} \in L^{1}(\mathbb{T})$ is the absolutely continuous part of $\mu$ and $\mu_{\perp}$ is the singular part, i.e., the support of $\mu_{\perp}$ has Lebesgue measure zero. Define the beam type functional

$$
\begin{gather*}
\phi: H \rightarrow[0,+\infty], \\
\phi(h):= \begin{cases}\int_{\mathbb{T}} e^{-\left(\left(\ln h_{x}\right)_{x}\right)_{\|}^{+}+\left(\left(\ln h_{x}\right)_{x}\right)^{-}} \mathrm{d} x, & \text { if } h \in V, \text { and }\left(\left(\ln h_{x}\right)_{x}\right)^{-} \ll \mathcal{L}^{1}, \\
+\infty & \text { otherwise. }\end{cases} \tag{2.4}
\end{gather*}
$$

Here $\left(\left(\ln h_{x}\right)_{x}\right)_{\|}$denotes the absolutely continuous part of $\left(\ln h_{x}\right)_{x},\left(\left(\ln h_{x}\right)_{x}\right)^{-}$is the negative part of $\left(\ln h_{x}\right)_{x}$ and $\left(\left(\ln h_{x}\right)_{x}\right)^{+}$is the positive part of $\left(\ln h_{x}\right)_{x}$ such that $\left(\left(\ln h_{x}\right)_{x}\right)^{ \pm}$are two nonnegative measures and $\left(\ln h_{x}\right)_{x}=\left(\left(\ln h_{x}\right)_{x}\right)^{+}-\left(\left(\ln h_{x}\right)_{x}\right)^{-}$. We call the singular part $\left(\left(\ln h_{x}\right)_{x}\right)_{\perp}^{+}$ latent singularity in solution $h$.

In view of the a priori estimate on the mass of the measure $h_{x x}$, we introduce the indicator function

$$
\psi: H \rightarrow\{0,+\infty\}, \quad \psi(h):= \begin{cases}0 & \text { if } h \in V,\left\|h_{x x}\right\|_{\mathcal{M}(\mathbb{T})} \leq C_{*},  \tag{2.5}\\ +\infty & \text { otherwise } .\end{cases}
$$

Here $C_{*}$ is a fixed constant, which is determined in (2.27) by the initial datum later.

### 2.2. Euler scheme

Even if (1.10) has a nice variational structure, and $V$ has Banach space structure. To avoid the technical difficulties brought by non-reflexivity we adopt the result [2, Theorem 4.0.4] by Ambrosio, Gigli and Savaré. After defining the energy functionals rigorously, the key process is to study the detailed properties of energy functionals. First let us establish the gradient flow evolution in the metric space ( $H$, dist), with distance dist $(u, v):=\|u-v\|_{H}$. Let $h_{0}(x) \in H$ be a given initial datum and $0<\tau \ll 1$ be a given parameter. We consider a sequence $\left\{x_{n}^{\tau}\right\}$ which satisfies the following unconditional-stable backward Euler scheme

$$
\left\{\begin{array}{l}
x_{n}^{(\tau)} \in \operatorname{argmin}_{x^{\prime} \in H}\left\{(\phi+\psi)\left(x^{\prime}\right)+\frac{1}{2 \tau}\left\|x^{\prime}-x_{n-1}^{(\tau)}\right\|_{H}^{2}\right\}, \quad n \geq 1  \tag{2.6}\\
x_{0}^{(\tau)}:=h_{0} \in H
\end{array}\right.
$$

The existence and uniqueness of the sequence $\left\{x_{n}^{\tau}\right\}$ can be proved by direct method in calculus of variation after we establishing the convexity and lower semi continuity of $\phi+\psi$ in Lemma 2.1; see also [8, Prop. 11]. Thus we consider the gradient descent with respect to $\phi+\psi$ in the space ( $H$, dist).

Now for any $0<\tau \ll 1$ we define the resolvent operator (see [2, p. 40])

$$
\mathcal{J}_{\tau}[h]:=\operatorname{argmin}_{v \in H}\left\{(\phi+\psi)(v)+\frac{1}{2 \tau}\|v-h\|_{H}^{2}\right\},
$$

then the variational approximation of $h$ at $t$ is obtained by Euler scheme (2.6) as

$$
\begin{equation*}
h_{n}(t):=\left(\mathcal{J}_{t / n}\right)^{n}\left[h_{0}\right] . \tag{2.7}
\end{equation*}
$$

In Proposition 2.4, we will use the theory for gradient flow in metric space [2, Theorem 4.0.4] to establish the convergence of the variational approximation $h_{n}(t)$ to variational inequality solution to (1.10), which is defined below.

Definition 1. Given initial data $h_{0} \in H$, we call $h:[0,+\infty) \rightarrow H$ a variational inequality solution to (1.10) if $h(t)$ is a locally absolutely continuous curve such that $\lim _{t \rightarrow 0} h(t)=h_{0}$ in $H$ and

$$
\begin{equation*}
\left\langle h_{t}(t), h(t)-v\right\rangle_{H^{\prime}, H} \leq \phi(v)-\phi(h(t)) \quad \text { for a.e. } t>0, \forall v \in D(\phi+\psi) . \tag{2.8}
\end{equation*}
$$

Next we study some properties, including convexity and lower semi continuity in $H$, of the functional $\phi+\psi$.

### 2.3. Convexity and lower semi continuity of function $\phi+\psi$ in $H$

Before we prove the convexity and lower semi continuity of function $\phi+\psi$, we first state an important lemma concerning the weak lower semi continuity of $\phi$ in BV space.

Proposition 2.1. Let $h_{n}, h \in V$. If $\left(\ln h_{n x}\right)_{x} \stackrel{*}{\rightharpoonup}\left(\ln h_{x}\right)_{x}$ in $\mathcal{M}(\mathbb{T})$, we have

$$
\begin{equation*}
\liminf _{n \rightarrow+\infty} \phi\left(h_{n}\right) \geq \phi(h) . \tag{2.9}
\end{equation*}
$$

Proof. Denote $\mu_{n}:=\left(\ln h_{n x}\right)_{x}, \mu:=\left(\ln h_{x}\right)_{x}$. Notice that $\phi$ defines only on the absolutely continuous part of $\left(\ln h_{x}\right)_{x}$. Since $\phi$ is convex functional by Lemma 2.2, so the inequality (2.9) is straightforward if the absolutely continuous part (resp. singular part) converges still to absolutely continuous part (resp. singular part). Thus the key point is to clarify the following two crossconvergence cases. (i) The absolutely continuous part of $\mu_{n}$ converges to the singular part of $\mu$; and (ii) the singular part of $\mu_{n}$ converges to the absolutely continuous part of $\mu$. For case (ii), notice $\phi$ is decreasing with respect to $\mu_{\|}$, which implies the limit $\phi(h)$ will be smaller with the additional absolutely continuous part. For case (i), delicate estimates using cut-off function is applied to show the loss in $\phi$ due to the additional singular part turns out to have a infinitely small contribution. We refer to [8, Proposition 5] for the detailed proofs of these two cases.

Next we will prove the convexity and lower semi continuity of function $\phi+\psi$ in $H$.
Lemma 2.2. The sum $\phi+\psi: H \rightarrow[0,+\infty]$ is proper, convex, lower semicontinuous in $H$ and satisfies coercivity defined in [2, (2.4.10)].

Proof. Clearly since the typical function $h=L x \in D(\phi+\psi)$, so $D(\phi+\psi)=\{\phi+\psi<+\infty\}$ is nonempty and $\phi+\psi$ is proper. Due to the positivity of $\phi, \psi$, coercivity [2, (2.4.10)], i.e., $\exists u * \in D(\phi+\psi), r *>0$ such that $\inf \{(\phi+\psi)(v): v \in H, \operatorname{dist}(v, u *) \leq r *\}>-\infty$, is obvious.

Convexity. Note that since both $\phi, \psi \geq 0$, we have $D(\phi+\psi)=D(\phi) \cap D(\psi)$. Given $u, v \in H$, $t \in(0,1)$, without loss of generality we assume $u, v \in D(\phi+\psi)$, otherwise convexity inequality is trivial. Therefore the measure $(1-t)\left(\ln u_{x}\right)_{x}+t\left(\ln v_{x}\right)_{x}$ has no negative singular part, while its positive singular part satisfies

$$
\left[(1-t)\left(\ln u_{x}\right)_{x}+t\left(\ln v_{x}\right)_{x}\right]_{\perp}^{+}=\left[(1-t)\left(\ln u_{x}\right)_{x}\right]_{\perp}^{+}+\left[t\left(\ln v_{x}\right)_{x}\right]_{\perp}^{+},
$$

and its absolutely continuous part satisfies

$$
\left[(1-t)\left(\ln u_{x}\right)_{x}+t\left(\ln v_{x}\right)_{x}\right]_{\|}=\left[(1-t)\left(\ln u_{x}\right)_{x}\right]_{\|}+\left[t\left(\ln v_{x}\right)_{x}\right]_{\|} .
$$

Thus we have

$$
\begin{aligned}
\phi((1-t) u+t v) & =\int_{\mathbb{T}} e^{-\left[\left(\ln \left[(1-t) u_{x}+t v_{x}\right]\right)_{x}\right]_{\|}} \mathrm{d} x \\
& \leq \int_{\mathbb{T}} e^{-\left[(1-t)\left(\ln u_{x}\right)_{x}+t\left(\ln v_{x}\right)_{x}\right]_{\|}} \mathrm{d} x \\
& =\int_{\mathbb{T}} e^{-(1-t)\left[\left(\ln u_{x}\right)_{x}\right]_{\|}-t\left[\left(\ln v_{x}\right)_{x}\right]_{\|}} \mathrm{d} x
\end{aligned}
$$

$$
\begin{aligned}
& \leq(1-t) \int_{\mathbb{T}} e^{-\left[\left(\ln u_{x}\right)_{x}\right]_{\|} \mathrm{d} x+t \int_{\mathbb{T}} e^{-\left[\left(\ln v_{x}\right)_{x}\right]_{\|}} \mathrm{d} x} \\
& =(1-t) \phi(u)+t \phi(v)
\end{aligned}
$$

where we used the convexity of $-\ln x$ and $e^{-x}$ in the two inequalities separately. Hence $\phi+\psi$ is convex.

Lower semicontinuity. Consider a sequence $h_{n} \rightarrow h$ in $H$. We need to check

$$
(\phi+\psi)(h) \leq \liminf _{n}(\phi+\psi)\left(h_{n}\right) .
$$

If $h_{n} \in D(\phi+\psi)$ does not hold for all large $n$, then lower semicontinuity holds. Without loss of generality, we can assume $h_{n} \in D(\phi+\psi)$ for all $n$, and also

$$
\liminf _{n}(\phi+\psi)\left(h_{n}\right)=\lim _{n}(\phi+\psi)\left(h_{n}\right) .
$$

First notice $h_{n} \in D(\phi)$ for any $n$ implies

$$
\begin{aligned}
\int_{\mathbb{T}}\left(\left(\ln h_{n x}\right)_{x}\right)^{-} \mathrm{d} x & \leq \int_{\mathbb{T} \cap\left(\left(\ln h_{n x}\right)_{x}\right)^{-}>0} e^{\left(\left(\ln h_{n x}\right)_{x}\right)^{-}} \mathrm{d} x \leq \int_{\mathbb{T}} e^{\left(\left(\ln h_{n x}\right)_{x}\right)^{-}-\left(\left(\ln h_{n x}\right)_{x}\right)_{\|}^{+}} \mathrm{d} x \\
& =\phi\left(h_{n}(t)\right) \leq C .
\end{aligned}
$$

Then similar to (1.16), we have

$$
\begin{equation*}
\frac{1}{2}\left\|\left(\ln h_{n x}\right)_{x}\right\|_{\mathcal{M}(\mathbb{T})}=\left\|\left(\left(\ln h_{n x}\right)_{x}\right)^{-}\right\|_{\mathcal{M}(\mathbb{T})}=\left\|\left(\left(\ln h_{n x}\right)_{x}\right)^{+}\right\|_{\mathcal{M}(\mathbb{T})} \leq C \tag{2.10}
\end{equation*}
$$

which yields that there exists $\mu \in \mathcal{M}(\mathbb{T})$ such that $\left(\ln h_{n x}\right)_{x} \stackrel{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\mathbb{T})$.
Second, since $h_{n} \in D(\psi)$, we have $\left\|h_{n x x}\right\|_{\mathcal{M}(\mathbb{T})} \leq C_{*}$. Thus strong convergence $h_{n} \rightarrow h$ in $H$, together with the embedding $B V(\mathbb{T}) \hookrightarrow L^{p}(\mathbb{T})$ compactly for any $p<\infty$, leads to the strong convergence $h_{n x} \rightarrow h_{x}$ in $L^{p}(\mathbb{T})$ for any $p<\infty$. Therefore we have $h_{n x} \rightarrow h_{x}$ almost everywhere and consequently $\ln h_{n x} \rightarrow \ln h_{x}$ almost everywhere. Combining this with $\left(\ln h_{n x}\right)_{x} \stackrel{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\mathbb{T})$ gives $\mu=\left(\ln h_{x}\right)_{x}$ and $\left(\ln h_{n x}\right)_{x} \stackrel{*}{ }\left(\ln h_{x}\right)_{x}$ in $\mathcal{M}(\mathbb{T})$.

Finally, since $h_{n x x} \stackrel{*}{\rightharpoonup} h_{x x}$ in $\mathcal{M}(\mathbb{T})$, we also know $h \in D(\psi)$ and $0=\psi\left(h_{n}\right)=\psi(h)$. Therefore by Proposition 2.1 we have

$$
\liminf _{n} \phi\left(h_{n}\right) \geq \phi(h)
$$

and the lower semicontinuity is proved.
As long as we have the convexity of $\phi+\psi$, the $\tau^{-1}$-convexity is standard and the proof can be found in [8, Lemma 10].

Proposition $2.3\left(\tau^{-1}\right.$-convexity). For any $h, v_{0}, v_{1} \in D(\phi+\psi)$, there exists a curve $v:[0,1] \rightarrow$ $D(\phi+\psi)$ such that $v(0)=v_{0}, v(1)=v_{1}$ and the functional

$$
\begin{equation*}
\Phi(\tau, h ; v):=(\phi+\psi)(v)+\frac{1}{2 \tau}\|h-v\|_{H}^{2} \tag{2.11}
\end{equation*}
$$

satisfies $\tau^{-1}$-convexity, i.e.,

$$
\begin{equation*}
\Phi(\tau, h ; v(t)) \leq(1-t) \Phi\left(\tau, h ; v_{0}\right)+t \Phi\left(\tau, h ; v_{1}\right)-\frac{1}{2 \tau} t(1-t)\left\|v_{0}-v_{1}\right\|_{H}^{2} \tag{2.12}
\end{equation*}
$$

for all $\tau>0, t \in[0,1]$.

### 2.4. Existence of variational inequality solution

After studying convexity and lower semicontinuity in last section, we shall apply the convergence result in [2, Theorem 4.0.4] to derive that the discrete solution $h_{n}$ obtained by Euler scheme (2.6) converges to the variational inequality solution defined in Definition 1. For $v \in D(\phi)$, denote the local slope

$$
\begin{equation*}
|\partial \phi|(v):=\limsup _{w \rightarrow v} \frac{\max \{\phi(v)-\phi(w), 0\}}{\operatorname{dist}(v, w)} . \tag{2.13}
\end{equation*}
$$

Proposition 2.4. Given $h_{0} \in H$, for any $t>0, t=n \tau$, let $h_{n}(t)$ defined in (2.7) be the approximation solution obtained by Euler scheme (2.6), then there exists a local Lipschitz curve $h(t):[0,+\infty) \rightarrow H$ such that

$$
\begin{equation*}
h_{n}(t) \rightarrow h(t) \text { in } L^{2}(\mathbb{T}) \tag{2.14}
\end{equation*}
$$

and $h:[0,+\infty) \rightarrow H$ is the unique EVI solution in the sense that $h$ is unique among all the locally absolutely continuous curves such that $\lim _{t \rightarrow 0} h(t)=h_{0}$ in $H$ and

$$
\begin{equation*}
\frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t}\|h(t)-v\|^{2} \leq(\phi+\psi)(v)-(\phi+\psi)(h(t)), \quad \text { a.e. } t>0, \forall v \in D(\phi+\psi) \tag{2.15}
\end{equation*}
$$

Moreover, we have the following regularities

$$
\begin{align*}
&(\phi+\psi)(h(t)) \leq(\phi+\psi)(v)+\frac{1}{2 t}\left\|v-h_{0}\right\|_{H}^{2}, \quad \forall v \in D(\phi+\psi)  \tag{2.16}\\
&|\partial(\phi+\psi)|^{2}(h(t)) \leq|\partial(\phi+\psi)|^{2}(v)+\frac{1}{t^{2}}\left\|v-h_{0}\right\|_{H}^{2}, \quad \forall v \in D(|\partial(\phi+\psi)|) . \tag{2.17}
\end{align*}
$$

This Proposition is a direct result by combining [2, Theorem 4.0.4] with Proposition 2.1 and Proposition 2.3. Other kinds of regularity estimates can also be obtained and we refer to [8, Theorem 13], [2, Theorem 4.0.4] for details. Next we claim the EVI solution obtained above is EVI solution to (1.10) with better properties as follows.

Theorem 2.5. Given any $T>0$ and initial datum $h_{0} \in H$ such that $\phi\left(h_{0}\right)<+\infty$,
(i) the solution obtained in Proposition 2.4 has the following regularities

$$
\begin{gathered}
h \in L^{\infty}([0, T] ; V) \cap C^{0}([0, T] ; H), \quad h_{t} \in L^{\infty}([0, T] ; H), \\
\left(\left(\ln h_{x}\right)_{x}\right)^{-} \ll \mathcal{L}^{1} \quad \text { for a.e. } t \in[0, T],
\end{gathered}
$$

where $\left(\left(\ln h_{x}\right)_{x}\right)^{-}$is the negative part of $\left(\ln h_{x}\right)_{x}$;
(ii) there exist constants $c_{1}, c_{2}>0$ depending only on $h_{0}$ and will be determined in (2.25) such that

$$
\begin{equation*}
c_{1} \leq h_{x} \leq c_{2} \tag{2.18}
\end{equation*}
$$

(iii) $h$ is the EVI solution in Definition 1, i.e.

$$
\begin{equation*}
\left\langle h_{t}(t), h(t)-v\right\rangle_{H^{\prime}, H} \leq \phi(v)-\phi(h(t)) \quad \text { for a.e. } t>0, \forall v \in D(\phi+\psi), \tag{2.19}
\end{equation*}
$$

and consequently we have the decay estimate

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} t} \int_{\mathbb{T}} h^{2} \mathrm{~d} x \leq 0 \tag{2.20}
\end{equation*}
$$

The dual pair $\langle\cdot, \cdot\rangle_{H^{\prime}, H}$ is the usual integration so we just use $\langle\cdot, \cdot\rangle$ in the following article. Recall the definition of $\phi$ in (2.4). $\phi\left(h_{0}\right)<+\infty$ if and only if $h_{0} \in V,\left(\left(\ln h_{x}^{0}\right)_{x}\right)^{-} \ll \mathcal{L}^{d}$ and $\int_{\mathbb{T}} e^{-\left(\left(\ln h_{x}^{0}\right)_{x}\right)_{\|}^{+}+\left(\left(\ln h_{x}^{0}\right)_{x}\right)^{-}} \mathrm{d} x<+\infty$.

Proof. First, we claim the functional $\psi$ can be taken off. Indeed, from (2.16) taking $v=h_{0}$ gives

$$
\begin{equation*}
(\phi+\psi)(h(t)) \leq(\phi+\psi)\left(h_{0}\right)<+\infty, \tag{2.21}
\end{equation*}
$$

which also implies

$$
\begin{equation*}
\phi(h(t)) \leq \phi\left(h_{0}\right)<+\infty \quad \text { for a.e. } t \in[0, T] . \tag{2.22}
\end{equation*}
$$

Now we use (2.22) to determine those constants $c_{1}, c_{2}$ in Theorem 2.5 and $C_{*}$ in Definition 2.5. Notice the screw periodic boundary condition (1.1), we have $\int_{\mathbb{T}} \mathrm{d}\left(h_{x x}\right)=0$, and then

$$
\begin{equation*}
\left\|\left(h_{x x}\right)^{+}\right\|_{\mathcal{M}(\mathbb{T})}=\left\|\left(h_{x x}\right)^{-}\right\|_{\mathcal{M}(\mathbb{T})}=\frac{1}{2}\left\|h_{x x}\right\|_{\mathcal{M}(\mathbb{T})} \tag{2.23}
\end{equation*}
$$

Thanks to

$$
\begin{aligned}
\int_{\mathbb{T}}\left(\left(\ln h_{x}\right)_{x}\right)^{-} \mathrm{d} x & \leq \int_{\mathbb{T} \cap\left(\left(\ln h_{x}\right)_{x}\right)^{-}>0} e^{\left(\left(\ln h_{x}\right)_{x}\right)^{-}} \mathrm{d} x \leq \int_{\mathbb{T}} e^{\left(\left(\ln h_{x}\right)_{x}\right)^{-}-\left(\left(\ln h_{x}\right)_{x}\right)_{\|}^{+}} \mathrm{d} x \\
& =\phi(h(t)) \leq \phi\left(h_{0}\right)
\end{aligned}
$$

we know $\left(\left(\ln h_{x}\right)_{x}\right)^{-} \ll \mathcal{L}^{1}$ for a.e. $t \in[0, T]$. Then similar to (2.23), we have

$$
\begin{equation*}
\frac{1}{2}\left\|\left(\ln h_{x}\right)_{x}\right\|_{\mathcal{M}(\mathbb{T})}=\left\|\left(\left(\ln h_{x}\right)_{x}\right)^{-}\right\|_{\mathcal{M}(\mathbb{T})}=\left\|\left(\left(\ln h_{x}\right)_{x}\right)^{+}\right\|_{\mathcal{M}(\mathbb{T})} \leq \phi\left(h_{0}\right) \tag{2.24}
\end{equation*}
$$

Due to the embedding $B V(\mathbb{T}) \hookrightarrow L^{\infty}(\mathbb{T})$ in one dimension, we have

$$
\left\|\ln h_{x}\right\|_{L^{\infty}(\mathbb{T})} \leq c \phi\left(h_{0}\right)
$$

which implies

$$
\begin{equation*}
c_{1}:=e^{-c \phi\left(h_{0}\right)} \leq h_{x} \leq e^{c \phi\left(h_{0}\right)}=: c_{2} \tag{2.25}
\end{equation*}
$$

and (ii). Combining (2.24) and (2.25), we conclude

$$
\begin{equation*}
\frac{1}{c_{1}}\left\|h_{x x}\right\|_{\mathcal{M}(\mathbb{T})} \leq\left\|\frac{h_{x x}}{h_{x}}\right\|_{\mathcal{M}(\mathbb{T})} \leq 2 \phi\left(h_{0}\right) \tag{2.26}
\end{equation*}
$$

Therefore in Definition (2.5), we can take

$$
\begin{equation*}
C_{*}:=2 c_{1} \phi\left(h_{0}\right)+1 \tag{2.27}
\end{equation*}
$$

and then

$$
\begin{equation*}
\psi(h(t)) \equiv 0 \equiv \partial \psi(h(t)) . \tag{2.28}
\end{equation*}
$$

The invariant ball introduced by indicate function $\psi$ is similar to the idea of a priori assumption method, i.e. we first obtain the solution in some invariant ball $\left\|h_{x x}\right\|_{\mathcal{M}} \leq C_{*}$, and then prove the invariant ball is not artificial by showing the solution always locates within the ball $\left\|h_{x x}\right\|_{\mathcal{M}} \leq$ $C_{*}-1$. Noticing also that if $v \in D(\psi), \psi(v)=0$, so EVI (2.15) is reduced to

$$
\frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t}\|h(t)-v\|^{2} \leq \phi(v)-\phi(h(t)) \quad \text { for a.e. } t>0, \forall v \in D(\phi+\psi) .
$$

Second, it remains to prove the $h_{t} \in L^{\infty}\left(0, T ; L^{2}(\mathbb{T})\right)$. From Theorem 2.4 we know that $t \mapsto h(t)$ is locally Lipschitz in $(0, T)$, i.e. for any $t_{0}>0$ there exists $M=M\left(t_{0}\right)>0$ such that

$$
\left\|h\left(t_{0}+\varepsilon\right)-h\left(t_{0}\right)\right\|_{L^{2}(\mathbb{T})} \leq M\left(t_{0}\right) \varepsilon \quad \text { for all } \varepsilon \in\left[0, T-t_{0}\right] .
$$

The key point is to obtain a uniform bound for $M\left(t_{0}\right)$ for arbitrary $t_{0} \geq 0$. By exactly the same argument in [8, Corollary 3.1] we can further show

$$
\begin{equation*}
\left\|h_{t}\right\|_{L^{\infty}\left(0, T ; L^{2}(\mathbb{T})\right)} \leq|\partial \phi|\left(h_{0}\right), \tag{2.29}
\end{equation*}
$$

which concludes (i).
Finally, from the regularity in (i) and

$$
\frac{1}{2} \frac{\mathrm{~d}}{\mathrm{~d} t}\|h(t)-v\|_{L^{2}(\mathbb{T})}^{2}=\left\langle h_{t}(t), h(t)-v\right\rangle
$$

we obtain (2.19). From (2.19), substituting $v=(1-\varepsilon) h$ for any $\varepsilon$ small enough shows

$$
\left\langle h_{t}(t), \varepsilon h(t)\right\rangle \leq \phi((1-\varepsilon) h)-\phi(h(t))=0,
$$

where we used a constant multiplier of $h$ does not change the value of $\phi$. This concludes (2.20).

## 3. Existence of strong solution

After establishing the regularity of variational inequality solution in Section 2.4, we start to prove that the variational inequality solution is also a strong solution. We first clarify the definition of strong solution in the sense that equation holds almost everywhere but has a latent singularity in the second derivatives. It is worth to mention that from Observation 2 the singularity in positive part of $\left(\ln h_{x}\right)_{x}$ does not destroy the evolution of the solution but it is not removable. Singular PDEs should be understood as a limit of some regularized problems. In our case, the singular PDE is understood in the sense that the equation holds almost everywhere after removing the singular part of $\left(\ln h_{x}\right)_{x}$.

Definition 2. Given initial datum $h_{0} \in H$ such that $\phi\left(h_{0}\right)<+\infty$, we call function

$$
h \in L^{\infty}([0, T] ; V) \cap C^{0}([0, T] ; H), \quad h_{t} \in L^{\infty}([0, T] ; H)
$$

a strong solution to (1.10) if $h$ satisfies

$$
\begin{equation*}
h_{t}=\left(\frac{1}{h_{x}}\left(e^{-\left(\left(\ln h_{x}\right)_{x}\right)_{\|}}\right)_{x}\right)_{x} \tag{3.1}
\end{equation*}
$$

for a.e. $(t, h) \in[0, T] \times \mathbb{T}$ with respect to Lebesgue measure, where $\left(\left(\ln h_{x}\right)_{x}\right)_{\|}$is the absolutely continuous part of $\left.\left(\ln h_{x}\right)_{x}\right)$ in the decomposition (2.3).

The definition of strong solution above follows the classical definition for strong solution to an abstract Cauchy problem described by the infinitesimal generator of a $C_{0}$-semigroup. However the equation holds almost everywhere after removing the singular part of $\left(\ln h_{x}\right)_{x}$. We clarify that the notion of strong solution is try to understand a singular PDE by removing some singular part, which does not destroy the dynamics over time. In other words, the singular PDE can be understood as a limit of a regularized problem for all time although there is a singularity at finite time. For instance, we are sure about the presence of a Dirac $\delta$-function (the singularity is not removable) but we try to find out a way to understand $e^{-\delta}$.

Since we have obtained the EVI solution, the idea is to prove the sub-differential of functional $\phi$ is single-valued by testing EVI (2.19) with $v:=h \pm \varepsilon \varphi$ for any function $\varphi \in C^{\infty}(\mathbb{T})$. Let us state the main existence theorem as follows.

Theorem 3.1. Given $T>0$, initial datum $h_{0} \in H$ such that $\phi\left(h_{0}\right)<+\infty$, then EVI solution $h$ obtained in Theorem 2.5 is also a strong solution to (1.10), i.e.,

$$
\begin{equation*}
h_{t}=\left(\frac{1}{h_{x}}\left(e^{-\left(\left(\ln h_{x}\right)_{x}\right)_{\|}}\right)_{x}\right)_{x} \tag{3.2}
\end{equation*}
$$

for a.e. $(t, h) \in[0, T] \times \mathbb{T}$ with respect to Lebesgue measure. Besides, we have the following dissipation inequality

$$
\begin{equation*}
\phi(h(t))=\int_{\mathbb{T}} e^{-\left(\left(\ln h_{x}\right)_{x}\right)_{\|}} \mathrm{d} x \leq \phi\left(h_{0}\right), \quad t \geq 0, \tag{3.3}
\end{equation*}
$$

where $\left(\left(\ln h_{x}\right)_{x}\right)_{\|}$is the absolutely continuous part of $\left.\left(\ln h_{x}\right)_{x}\right)$ in the decomposition (2.3).
Proof. The general idea is to characterize the sub-differential of functional $\phi$ by testing EVI (2.19) with $v:=h \pm \varepsilon \varphi$ for any function $\varphi \in C^{\infty}(\mathbb{T})$ and then taking limit $\varepsilon \rightarrow 0$. In order to pass the limit, we should first obtain some integrability results in Step 1 and then calculate the sub-differential of $\phi$ in Step 2 . Here we only prove the integrability for some dense subset of $C^{\infty}(\mathbb{T})$ which will be enough for the calculation in Step 2. Indeed, we will take advantage that if we can obtain Gâteaux-derivative on some dense subset then the sub-differential is single-valued and equals the Gâteaux-derivative.

Step 1. Integrability results.
Assume $h(t)$ is EVI solution obtained in Theorem 2.5. To ensure we can take limit after testing EVI with $v:=h \pm \varepsilon \varphi$, we need to prove

$$
\begin{equation*}
e^{-\left(\left(\ln h_{x}\right)_{x}\right)_{\|}} \in L^{1}(\mathbb{T}) \tag{3.4}
\end{equation*}
$$

and

$$
\begin{equation*}
e^{-\left[\left(\ln \left(h_{x}+\varepsilon \varphi_{x}\right)\right)_{x}\right]_{\|}} \in L^{1}(\mathbb{T}) \tag{3.5}
\end{equation*}
$$

for $\varepsilon$ small enough and $\varphi$ in some dense set of $C_{b}^{\infty}(\mathbb{T})$. First from (2.22) we know $\phi(h(t)) \leq$ $\phi\left(h_{0}\right)$, which gives (3.3) and (3.4).

Next, we prove (3.5) for $\varphi$ in some dense set of $C_{b}^{\infty}(\mathbb{T})$. For any $c>0$, define

$$
\begin{equation*}
D_{c}:=\left\{\varphi \in C_{b}^{\infty}(\mathbb{T}) ;\left|\left(h_{x x}\right)_{\|} \varphi_{x}\right| \leq c\right\} ; \quad D:=\cup_{c \geq 0} D_{c} . \tag{3.6}
\end{equation*}
$$

We claim the set $D$ is dense in $L^{\infty}(\mathbb{T})$. Indeed, for any $\varphi \in L^{\infty}(\mathbb{T})$ define

$$
\varphi_{n}:= \begin{cases}\varphi & \text { if }\left|\left(h_{x x}\right)_{\|} \varphi_{x}\right| \leq n \\ 0 & \text { otherwise }\end{cases}
$$

Then for any $1 \leq p<\infty$,

$$
\left\|\varphi_{n}-\varphi\right\|_{L^{p}(\mathbb{T})}=\left(\int_{\left\{\mid\left(h_{x x}\right)_{\left.\| \varphi_{x} \mid>n\right\}}\right.}\left|\varphi_{n}-\varphi\right|^{p} \mathrm{~d} x\right)^{\frac{1}{p}}
$$

Since $\left|\left(h_{x x}\right)_{\|} \varphi_{x}\right| \leq\left|\left(h_{x x}\right)\|\mid\| \varphi_{x} \|_{L^{\infty}}\right.$,

$$
\left\{\left|\left(h_{x x}\right)_{\|} \varphi_{x}\right|>n\right\} \subseteq\left\{\mid\left(h_{x x}\right)_{\|}\| \| \varphi_{x} \|_{L^{\infty}}>n\right\}
$$

Therefore

$$
\begin{aligned}
\left\|\varphi_{n}-\varphi\right\|_{L^{p}(\mathbb{T})} & \leq\left(\int_{\left\{\left|\left(h_{x x}\right)_{\|}\right|>\frac{n}{\left\|\varphi_{x}\right\|_{L^{\infty}}}\right\}}\left|\varphi_{n}-\varphi\right|^{p} \mathrm{~d} x\right)^{\frac{1}{p}} \\
& \leq\|\varphi\|_{L^{\infty}}\left|\left\{\left|\left(h_{x x}\right)_{\|}\right|>\frac{n}{\left\|\varphi_{x}\right\|_{L^{\infty}}}\right\}\right|^{\frac{1}{p}} \rightarrow 0
\end{aligned}
$$

as $n \rightarrow \infty$, where we used the integrability $\left(h_{x x}\right)_{\|} \in L^{1}(\mathbb{T})$. Therefore we know $D$ is dense in $L^{\infty}$ and thus $D$ is dense in $C_{b}^{\infty}(\mathbb{T})$.

For any $\varphi \in D,\left|\varphi_{x x}\right| \leq\left\|\varphi_{x x}\right\|_{L^{\infty}}$ and there exists some $c^{0}$ such that $\left|\left(h_{x x}\right)_{\|} \varphi_{x}\right| \leq c^{0}$. Notice also $c_{1} \leq h_{x} \leq c_{2}$ due to (2.18). Hence for $\varepsilon$ small enough,

$$
\begin{aligned}
& \int_{\mathbb{T}} e^{-\left(\left(\ln \left(h_{x}+\varepsilon \varphi_{x}\right)\right)_{x}\right)_{\|}} \mathrm{d} x=\int_{\mathbb{T}} e^{-\left[\frac{h_{x x}+\varepsilon \varphi_{x x}}{h_{x}+\varepsilon \varphi_{x}}\right]_{\|}} \mathrm{d} x \\
= & \int_{\mathbb{T}} e^{-\frac{\left(h_{x x}\right)_{\|}}{h_{x}+\varepsilon \varphi_{x}}} e^{-\varepsilon \frac{\varphi_{x x}}{h_{x}+\varepsilon \varphi_{x}}} \mathrm{~d} x \\
\leq & C\left(c_{1}, c_{2},\|\varphi\|_{W^{2, \infty}}\right)\left[\int_{\left(\left(h_{x x}\right) \|^{+}\right)^{+}>0} e^{-\frac{\left(h_{x x}\right)_{\|}}{h_{x}+\varepsilon \varphi_{x}}} \mathrm{~d} x+\int_{\left(\left(h_{x x}\right)_{\|}\right)^{+}=0} e^{-\frac{\left(h_{x x}\right)_{\|}}{h_{x}+\varepsilon \varphi_{x}}} \mathrm{~d} x\right] \\
\leq & C\left(c_{1}, c_{2},\|\varphi\|_{W^{2, \infty}}\right)+C\left(c_{1}, c_{2},\|\varphi\|_{\left.W^{2, \infty}\right)} \int_{\left(\left(h_{x x}\right)_{\|}\right)^{+}=0} e^{-\frac{\left(h_{x x}\right)_{\|}}{h_{x}}\left(1+\varepsilon \frac{\left|\varphi_{x}\right|}{h_{x}}\right)} \mathrm{d} x .\right.
\end{aligned}
$$

Here for the first term in the last inequality we used $e^{-\tau}<1$ for $\tau>0$. For the second term in the last inequality, we used when $\left(\left(h_{x x}\right)_{\|}\right)^{+}=0$,

$$
-\frac{\left(h_{x x}\right)_{\|}}{h_{x}} \frac{1}{1+\varepsilon \frac{\varphi_{x}}{h_{x}}} \leq-\frac{\left(h_{x x}\right)_{\|}}{h_{x}}\left(1+2 \varepsilon \frac{\left|\varphi_{x}\right|}{h_{x}}\right)
$$

due to $\frac{1}{1+\varepsilon y} \leq 1+2 \varepsilon|y|$ for any $\varepsilon<\frac{1}{2 \max |y|}$. Therefore

$$
\begin{aligned}
& \int_{\mathbb{T}} e^{-\left(\left(\ln \left(h_{x}+\varepsilon \varphi_{x}\right)\right)_{x}\right)_{\|}} \mathrm{d} x \\
\leq & C\left(c_{1}, c_{2},\|\varphi\|_{W^{2}, \infty}\right)+C\left(c_{1}, c_{2},\|\varphi\|_{W^{2, \infty}}\right) \int_{\left(\left(h_{x x}\right) \|^{+}=0\right.} e^{-\frac{\left(h_{x x}\right) \|}{h_{x}}\left(1+2 \varepsilon \frac{\left|\varphi_{x}\right|}{h_{x}}\right)} \mathrm{d} x
\end{aligned}
$$

$$
\begin{aligned}
& \leq C\left(c_{1}, c_{2},\|\varphi\|_{W^{2, \infty}}\right)+C\left(c_{1}, c_{2},\|\varphi\|_{W^{2, \infty}}\right) \int_{\left(\left(h_{x x}\right)_{\|} \|^{+}=0\right.} e^{-\frac{\left(h_{x x}\right)_{\|}}{h_{x}}} e^{-2 \varepsilon \frac{\left(h_{x x}\right)_{\|} \varphi_{x} \mid}{h_{x}^{x}}} \mathrm{~d} x \\
& \leq C\left(c_{1}, c_{2},\|\varphi\|_{W^{2, \infty}}\right)+C\left(c^{0}, c_{1}, c_{2},\|\varphi\|_{W^{2, \infty}}\right) \int_{\mathbb{T}} e^{-\frac{\left(h_{x x}\right)}{h_{x}}} \mathrm{~d} x
\end{aligned}
$$

where we used $\left|\left(h_{x x}\right)_{\|} \varphi_{x}\right| \leq c^{0}$ in the last inequality and $C\left(c^{0}, c_{1}, c_{2},\|\varphi\|_{W^{2, \infty}}\right)$ is a generic constant depending only on $c^{0}, c_{1}, c_{2},\|\varphi\|_{W^{2, \infty}}$. This, together with (3.4) leads to (3.5).

## Step 2. Testing (2.19) with $v=h \pm \varepsilon \varphi$.

First we show $v \in D(\phi+\psi)$. Since $\varphi \in C_{b}^{\infty}(\mathbb{T})$ and (2.18), we can choose $\varepsilon$ small enough such that $h_{x}+\varepsilon \varphi_{x}>0$, so by definition of $\phi$ and $\left(h_{x x}+\varepsilon \varphi_{x x}\right)^{-} \in L^{1}$ we have $v \in D(\phi)$. It is sufficient to show $v \in D(\psi)$ for $\varepsilon$ small enough. Indeed, from (2.26) we know $\left\|h_{x x}\right\|_{\mathcal{M}} \leq$ $2 c_{1} \phi\left(h_{0}\right)=C_{*}-1$. Hence we choose $\varepsilon$ small enough such that $\varepsilon \leq \frac{1}{2\|\varphi\|_{W^{2, \infty}}}$, which implies $\|v\|_{\mathcal{M}} \leq 2 c_{1} \phi\left(h_{0}\right)+\frac{1}{2}<C_{*}$ and $\psi(v)=0$.

Plugging $v=h+\varepsilon \varphi$ into (2.19) gives

$$
\begin{align*}
0 & \leq\left\langle h_{t}(t), \varepsilon \varphi\right\rangle+\phi(h(t)+\varepsilon \varphi)-\phi(h(t)) \\
& =\varepsilon\left\langle h_{t}(t), \varphi\right\rangle+\int_{\mathbb{T}} e^{-\left[\frac{h_{x x}+\varepsilon \varphi_{x x}}{h_{x}+\varepsilon \varphi_{x}}\right]_{\|}}-e^{-\frac{\left(h_{x x}\right)_{\|}}{h_{x}}} \mathrm{~d} x . \tag{3.7}
\end{align*}
$$

We divide this by $\varepsilon>0$ and take limit $\varepsilon \rightarrow 0^{+}$. Thanks to the dominated convergence theorem and the integrability (3.5), we just need to check the pointwise limit for the integrand in the dense set $D$. Notice again $c_{1} \leq h_{x} \leq c_{2}$ due to (2.18) and thus $\frac{\varphi_{x}}{h_{x}}$ are uniformly bounded. For any $x \in \mathbb{T}, \varphi \in D$, we have

$$
\begin{aligned}
& \frac{1}{\varepsilon}\left[e^{-\left(\frac{h_{x x}+\varepsilon \varphi_{x x}}{h_{x}+\varepsilon \varphi_{x}}\right)_{\|}}-e^{-\frac{\left(h_{x x}\right)_{\|}}{h_{x}}}\right] \\
& =\frac{1}{\varepsilon}\left[e^{-\frac{\left(h_{x x}\right)_{\|}+\varepsilon \varphi_{x x}}{h_{x}\left(1+\varepsilon \varphi_{x}\right.} h_{x}}-e^{-\frac{\left(h_{x x}\right)_{\|}}{h_{x}}}\right] \\
& =\frac{1}{\varepsilon}\left[e^{-\frac{\left(h_{x x}\right)_{\|}+\varepsilon \varphi_{x x}}{h_{x}}\left[1-\varepsilon \frac{\varphi_{x}}{h_{x}}+O\left(\varepsilon^{2}\right)\right]}-e^{-\frac{\left(h_{x x}\right)_{\|}}{h_{x}}}\right] \\
& =\frac{1}{\varepsilon}\left[e^{-\frac{\left(h_{x x}\right)_{\|}}{h_{x}}-\varepsilon \frac{\varphi_{x x}}{h_{x}}+\varepsilon \frac{\left(h_{x x}\right) \|}{h_{x}} \frac{\varphi_{x}}{h_{x}}+O\left(\varepsilon^{2}\right)}-e^{-\frac{\left(h_{x x}\right)_{\|}}{h_{x}}}\right] \\
& =\frac{1}{\varepsilon} e^{-\frac{\left(h_{x x}\right)_{\|}}{h_{x}}-\varepsilon \frac{\varphi_{x x}}{h_{x}}+\varepsilon \frac{\left(h_{x x}\right) \|}{h_{x}} \frac{\varphi_{x}}{h_{x}}+O\left(\varepsilon^{2}\right)}\left[1-e^{\varepsilon \frac{\varphi_{x x}}{h_{x}}-\varepsilon \frac{\left(h_{x x}\right) \|}{h_{x}} \frac{\varphi_{x}}{h_{x}}+O\left(\varepsilon^{2}\right)}\right] \\
& \leq \frac{1}{\varepsilon} e^{-\frac{\left(h_{x x}\right)_{\|}}{h_{x}}-\varepsilon \frac{\varphi_{x x}}{h_{x}}+\varepsilon \frac{\left(h_{x x}\right)}{h_{x}} \frac{\varphi_{x}}{h_{x}}+O\left(\varepsilon^{2}\right)}\left[-\varepsilon \frac{\varphi_{x x}}{h_{x}}+\varepsilon \frac{\left(h_{x x}\right)_{\|}}{h_{x}} \frac{\varphi_{x}}{h_{x}}+O\left(\varepsilon^{2}\right)\right] \\
& \rightarrow e^{-\frac{\left(h_{x x}\right) \|}{h_{x}}}\left[-\frac{\varphi_{x x}}{h_{x}}+\frac{\left(h_{x x}\right)_{\|}}{h_{x}} \frac{\varphi_{x}}{h_{x}}\right]
\end{aligned}
$$

as $\varepsilon \rightarrow 0^{+}$, where we used $1-e^{x} \leq-x$ for all $x \in \mathbb{R}$ in the inequality. Then taking limit in (3.7) yields

$$
\begin{aligned}
& \left\langle h_{t}, \varphi\right\rangle+\lim _{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{\mathbb{T}}\left[e^{\left.-\left[\frac{h_{x x}+\varepsilon \varphi_{x x}}{h_{x}+\varepsilon \varphi_{x}}\right]_{\|}-e^{-\frac{\left(h_{x x}\right)_{\|}}{h_{x}}}\right] \mathrm{d} x}\right. \\
= & \left\langle h_{t}, \varphi\right\rangle+\int_{\mathbb{T}} e^{-\frac{\left(h_{x x}\right)_{\|}}{h_{x}}}\left[-\frac{\varphi_{x x}}{h_{x}}+\frac{\left(h_{x x}\right)_{\|}}{h_{x}} \frac{\varphi_{x}}{h_{x}}\right] \mathrm{d} x \geq 0
\end{aligned}
$$

for any $\varphi \in D$. Repeating the above arguments with $v=h-\varepsilon \varphi$ gives

$$
\left\langle h_{t}, \varphi\right\rangle+\int_{\mathbb{T}} e^{-\frac{\left(h_{x x}\right)}{h_{x}}}\left[-\frac{\varphi_{x x}}{h_{x}}+\frac{\left(h_{x x}\right)_{\|}}{h_{x}} \frac{\varphi_{x}}{h_{x}}\right] \mathrm{d} x \leq 0 .
$$

Then we finally obtain

$$
\begin{equation*}
\int_{\mathbb{T}} h_{t} \varphi+e^{-\frac{\left(h_{x x}\right)_{\|}}{h_{x}}}\left[-\frac{\varphi_{x x}}{h_{x}}+\frac{\left(h_{x x}\right)_{\|}}{h_{x}} \frac{\varphi_{x}}{h_{x}}\right] \mathrm{d} x=0 . \tag{3.8}
\end{equation*}
$$

By the dense argument for Gâteaux-derivative, this equality holds for any $\varphi \in C_{b}^{\infty}(\mathbb{T})$.
Now we integrate by parts for $\int_{\mathbb{T}}-\frac{1}{h_{x}} e^{-\frac{\left(h_{x x}\right) \|}{h_{x}}} \varphi_{x x} \mathrm{~d} x$. By the Radon-Nikodym theorem, (3.8) is reduced to

$$
\int_{\mathbb{T}} h_{t} \varphi+\left[\frac{1}{h_{x}}\left[\left(e^{-\frac{\left(h_{x x}\right)_{\|}}{h_{x}}}\right)_{x}\right]_{\|}+\frac{1}{h_{x}}\left[\left(e^{-\frac{\left(h_{x x}\right)_{\|}}{h_{x}}}\right)_{x}\right]_{\perp}-\frac{\left(h_{x x}\right)_{\perp}}{h_{x}^{2}} e^{-\frac{\left(h_{x x}\right)_{\|}}{h_{x}}}\right] \varphi_{x} \mathrm{~d} x=0 .
$$

Therefore

$$
h_{t}-\left[\frac{1}{h_{x}}\left[\left(e^{-\frac{\left(h_{x x}\right) \|}{h_{x}}}\right)_{x}\right]_{\|}+\frac{1}{h_{x}}\left[\left(e^{-\frac{\left(h_{x x}\right) \|}{h_{x}}}\right)_{x}\right]_{\perp}-\frac{\left(h_{x x}\right)_{\perp}}{h_{x}^{2}} e^{-\frac{\left(h_{x x}\right)_{\|}}{h_{x}}}\right]_{x}=0
$$

in $\left(C_{b}^{\infty}(\mathbb{T})\right)^{\prime}$, which leads to

$$
h_{t}-\left[\frac{1}{h_{x}}\left(e^{-\frac{\left(h_{x x}\right)_{\|}}{h_{x}}}\right)_{x}\right]_{x}=0
$$

for a.e. $(t, x) \in[0, T] \times \mathbb{T}$ with respect to Lebesgue measure and concludes $h$ is a strong solution.

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