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PAPER

Homogenisation of Wasserstein gradient flows

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Abstract
We prove the convergence of a Wasserstein gradient flow of a free energy in inhomogeneous media. Both the energy and media can depend on the spatial variable in a fast oscillatory manner. In particular, we show that the gradient-flow structure is preserved in the limit, which is expressed in terms of an effective energy and Wasserstein metric. The gradient flow and its limiting behavior are analysed through an energy dissipation inequality. The result is consistent with asymptotic analysis in the realm of homogenisation. However, we note that the effective metric is in general different from that obtained from the Gromov–Hausdorff convergence of metric spaces. We apply our framework to a linear Fokker–Planck equation, but we believe the approach is robust enough to be applicable in a broader context.

1. Introduction

Optimal transport has appeared in many practical and theoretical applications, cf. [41, 43, 44, 52, 53]. Precisely, given a cost function $c(\cdot,\cdot):\mathbb{R}^n\times\mathbb{R}^n\longrightarrow\mathbb{R}$, and two probability measures μ,ν on $\mathbb{R}^n$, the problem of optimal transport is to find the minimum cost of transporting μ to ν. It has the following two classical formulations: first by Monge [39] in terms of optimal transport map and a second formulation using duality by Kantorovich [33] in terms of optimal coupling measure:

$$\text{Monge:}\qquad \inf\left\{\int c(x,\Phi(x))\,\mathrm{d}\mu(x):\ \Phi:\mathbb{R}^n\longrightarrow\mathbb{R}^n,\quad \Phi_\sharp\mu=\nu\right\},$$

and

$$\text{Kantorovich:}\qquad \inf\left\{\iint c(x,y)\,\mathrm{d}\gamma(x,y);\quad \int\gamma(x,\,\mathrm{d}y)=\mu(x),\ \int\gamma(\,\mathrm{d}x,y)=\nu(y)\right\}.$$

In the above, γ is a probability measure on the product space $\mathbb{R}^n\times\mathbb{R}^n$. The equivalence of the above, under appropriate general assumptions, has been established in ref. [42]. Typical examples of cost functions include the Euclidean distance square, $c(x,y)=|x-y|^2$ which is convex and spatially homogeneous in the sense that $c(x,y)=c(x-y)$. In this case, the infimum value of the above two formulations is the square of Wasserstein-2 distance between μ and ν, denoted as $W_2^2(\mu,\nu)$. We refer to [2, 46, 52, 53] for examples of monographs on the theory of optimal transports.

The main purpose of the current paper is to incorporate spatial inhomogeneity into the above problem, or more precisely, the cost function c. We then consider gradient flows with respect to the Wasserstein metric induced by c and analyse their limiting behaviour or description when the inhomogeneity converges in appropriate sense. We believe these types of questions appear naturally in many applications such as urban transportations [8, 11], network science [32], spread of epidemics [7], optics [45], and many others. Such a consideration indeed has a long history in the realm of homogenisation [10, 48].

On a technical level, we aim to explore how the ideas of homogenisation can be introduced into optimal transport problems. Even though in the current paper, we work in a spatially continuous setting, the problem formulation can be posed in a discrete, graph or network setting, as seen from the above-mentioned applications. See also the end of this section for some mathematical work on these attempts.

To be specific, we consider cost functions $c_\varepsilon(\cdot,\cdot)$ that depend on the spatial variables in some oscillatory manner. We find that the formulation of Benamou–Brenier [5] is well-suited for this purpose. Not only does it connect optimal transport to some underlying "dynamical process," it allows us to incorporate spatial inhomogeneity "more or less at will". More precisely, we focus on the case that $c_\varepsilon(x,y)$ is defined through a *least action principle*,

$$c_\varepsilon(x,y) = \min\left\{\int_0^1 L_\varepsilon(\dot{z}_t, z_t)\,\mathrm{d}t, \quad z:[0,1]\longrightarrow\mathbb{R}^n,\ z_0 = x,\ z_1 = y\right\}, \tag{1.1}$$

where we envision that L_ε is convex in the first variable $v = \dot{z}_t$ and oscillatory or periodic in the second variable z_t. Note that this cost function also defines a metric in an inhomogeneous media with periodic structure. If one further assumes that L is a bilinear form in v, given by a positive definite matrix $B_\varepsilon(x)$,

$$L(v,z) = \langle B_\varepsilon(z)v,\ v\rangle, \tag{1.2}$$

then $c_\varepsilon(x,y)$ defines a Riemannian metric on $\mathbb{R}^n$

$$c_\varepsilon^2(x,y) = \min\left\{\int_0^1 \langle B_\varepsilon(z_t)\dot{z}_t, \dot{z}_t\rangle\,\mathrm{d}t, \quad z:[0,1]\longrightarrow\mathbb{R}^n,\ z_0 = x,\ z_1 = y\right\}. \tag{1.3}$$

The above leads to the following ε-Wasserstein distance (square) between $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$,

$$W_\varepsilon^2(\mu,\nu) := \inf\left\{\iint c_\varepsilon(x,y)\,\mathrm{d}\gamma(x,y); \quad \int \gamma(x,\ \mathrm{d}y) = \mu(x),\ \int \gamma(\,\mathrm{d}x, y) = \nu(y)\right\}. \tag{1.4}$$

The description and formulation in this and next sections is applicable for general spatially inhomogeneous B_ε, but the focus of this paper is when B_ε takes the form $B_\varepsilon(x) = B\left(\frac{x}{\varepsilon}\right)$ – see Section 2.4 for precise statements and assumptions.

In order to keep the technicality in this paper manageable, we will only consider probability measures having densities with respect to the Lebesgue measure. Henceforth, for simplicity, we will use $\mathcal{P}_2(\mathbb{R}^n)$ to denote these measures or their densities. The subscript 2 means these measures have finite second moments. More precise assumptions will be stated in Section 2.4. Now let $(\mathcal{P}_2(\mathbb{R}^n), W_\varepsilon)$ be the Polish space endowed with the ε-Wasserstein metric. The main questions we want to understand are: *whether gradient-flow structures in $(\mathcal{P}_2(\mathbb{R}^n), W_\varepsilon)$ are preserved as $\varepsilon \to 0$ and if so, what the limiting Wasserstein distance $\overline{W}$ and gradient flow are.* We have given positive results for the case of linear Fokker–Planck equations in periodic media.

With (1.3), the ε-Wasserstein distance W_ε can be expressed using the following spatially inhomogeneous Benamou–Brenier formulation,

$$W_\varepsilon^2(\rho_0,\rho_1) := \inf\left\{\int_0^1\int \rho_t(x)\langle B_\varepsilon(x)v_t(x), v_t(x)\rangle\,dx\,dt, \quad (\rho_t, v_t)\in V(\rho_0,\rho_1)\right\} \tag{1.5}$$

where

$$V(\rho_0,\rho_1) := \left\{(\rho_t, v_t): \frac{\partial\rho_t}{\partial t} + \nabla\cdot(\rho_t v_t) = 0, \quad \rho(\cdot,0) = \rho_0, \quad \rho(\cdot,1) = \rho_1\right\}. \tag{1.6}$$

The work [6] – see its Theorems A and B – in fact shows that the inf of (1.5) (and (1.4)) is achieved by a unique interpolation between ρ_0 and ρ_1, given by a flow map $\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^\varepsilon = v_t(\Phi_t^\varepsilon)$,

$$\rho_t = (\Phi_t^\varepsilon)_\sharp\rho_0, \quad 0 \le t \le 1. \tag{1.7}$$

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Note that for the case $\varepsilon = 1, B_\varepsilon = I$, (1.5) is the celebrated Benamou–Brenier formula [5] for the standard (squared) Wasserstein distance

$$W_2^2(\rho_0, \rho_1) = \inf \left\{ \iint |x-y|^2 \,\mathrm{d}\gamma(x,y); \quad \int \gamma(x, \,\mathrm{d}y) = \rho_0(x)\,\mathrm{d}x, \;\int \gamma(\,\mathrm{d}x, y) = \rho_1(y)\,\mathrm{d}y \right\}. \tag{1.8}$$

The functional in (1.5) defines an action functional on $(\mathcal{P}_2(\mathbb{R}^n), W_2)$, which allows one to directly use least action principles on $(\mathcal{P}_2(\mathbb{R}^n), W_2)$ to compute the W_2-distance. In the seminal paper [40], Otto went further to regard W_2 as a Pseudo-Riemannian distance on $\mathcal{P}_2(\mathbb{R}^n)$ with the Riemannian metric being the same as the one given by the Benamou–Brenier formula. More precisely, for any s_1, s_2 on the tangent plane $T_\mathcal{P}$ at $\rho \in \mathcal{P}$, the metric tensor on $T_\mathcal{P} \times T_\mathcal{P}$ is given by

$$\langle s_1, s_2 \rangle_{T_\mathcal{P}, T_\mathcal{P}} := \int \rho(x) \langle \nabla\varphi_1(x), \nabla\varphi_2(x)\rangle \, dx, \qquad \text{where } s_i = -\nabla \cdot (\rho \nabla \varphi_i), \quad i = 1, 2. \tag{1.9}$$

(See Section 2.2 for an explanation of going from v_t in (1.5) to $\nabla\varphi$ above.) With the above set-up for the Wasserstein distance, we proceed to consider gradient flows in $(\mathcal{P}_2(\mathbb{R}^n), W_\varepsilon)$ of a given energy functional $E_\varepsilon : \mathcal{P}_2(\mathbb{R}^n) \longrightarrow \mathbb{R}$,

$$\partial_t \rho_t^\varepsilon = -\nabla^{W_\varepsilon} E_\varepsilon(\rho_t^\varepsilon). \tag{1.10}$$

The precise dynamics is uniquely determined by a dissipation functional on the tangent plane characterising the rate of change of the energy from which the Wasserstein gradient ∇^{W_ε} is derived. In this paper, we consider energy dissipation expressed by the metric W_ε (induced by (1.5)). It turns out W_ε can be formally interpreted as a Riemannian metric (see (2.11)), which in particular is given by a bilinear form. Based on the expression of ∇^{W_ε} (see (2.14)), ε-Wasserstein gradient flow (1.10) can be explicitly written as

$$\partial_t \rho_t^\varepsilon = \nabla \cdot \left(\rho_t^\varepsilon B_\varepsilon^{-1} \nabla \frac{\delta E_\varepsilon}{\delta \rho}(\rho_t^\varepsilon) \right). \tag{1.11}$$

Note that our formulation allows oscillations in both the energy E_ε and media B_ε.

If the total energy is taken as the relative entropy or the Kullback–Leibler divergence between ρ and another probability distribution $\pi_\varepsilon \in \mathcal{P}_2(\mathbb{R}^n)$,

$$E_\varepsilon(\rho) = KL(\rho || \pi_\varepsilon) := \int_{\mathbb{R}^n} \rho(x) \log \frac{\rho(x)}{\pi_\varepsilon(x)} \,\mathrm{d}x, \tag{1.12}$$

then the above ε-Wasserstein gradient flow (1.11) is the same as a linear Fokker–Planck equation with oscillatory coefficients. The above energy is often called the free energy of the system and π_ε in (1.12) is a stationary distribution corresponding to an underlying stochastic process.

Our main result is the evolutionary convergence of the ε-Wasserstein gradient flow (1.11) as $\varepsilon \to 0$, to a limit also characterised as a gradient flow of an effective total energy $\overline{E}$ with respect to an effective Wasserstein distance $\overline{W}$. The distance $\overline{W}$ induced by the evolutionary convergence is still a Riemannian metric on $\mathcal{P}_2(\mathbb{R}^n)$. However, we find that it is in general different from the direct Gromov–Hausdorff limit of W_ε. Even though our main result is proven for continuous state spaces, the approach we used for proving the convergence of multi-scale gradient flows can also be applied to discrete state spaces, in particular, graphs with inhomogeneous structure.

The main approach we use is to first recast the ε-Wasserstein gradient flow (1.11) as a generalised gradient flow in the following form of an energy dissipation inequality (EDI)

$$E_\varepsilon(\rho_t^\varepsilon) + \int_0^t \left[\psi_\varepsilon(\rho_\tau^\varepsilon, \partial_\tau \rho_\tau^\varepsilon) + \psi_\varepsilon^* \left(\rho_\tau^\varepsilon, -\frac{\delta E_\varepsilon}{\delta\rho}(\rho_\tau^\varepsilon) \right) \right] \mathrm{d}\tau \le E_\varepsilon(\rho_0^\varepsilon). \tag{1.13}$$

This formulation involves dissipation functionals ψ_ε and ψ_ε^* on the tangent and the co-tangent plane of $\mathcal{P}_2(\mathbb{R}^n)$, respectively. Inequality (1.13) is in fact equivalent to the strong form of gradient flow (1.10) since the functional ψ_ε and ψ_ε^* are convex conjugate of each other; for details, see Section 2.3. Then the limiting behaviour of the dynamics is obtained by considering the limit of the functionals in (1.13).

The framework using the EDI formulation of gradient flows to obtain the evolutionary Γ-convergence of gradient flows was first established by Sandier and Serfaty [49, 47]. In this setting, the key estimates are the lower bounds of the free energy and the energy dissipations in terms of the metric velocity and the metric slope. Many generalisations of the evolutionary convergence for generalised gradient flow systems are developed by Mielke, Peletier and collaborators; see the concept of energy-dissipation-principle (EDP) convergence of gradient flows in [4, 34], the concept of generalised tilt/contact EDP convergence developed in [16, 38], and also the review [36].

Following the above general framework for evolutionary Γ-convergence of gradient flows, we pass the limit in ε-EDI (1.13) by proving the lower bounds of all three functionals on the left-hand-side of (1.13): the energy functional E_ε, the time integrals of dissipation functionals ψ_ε and ψ_ε^*. The lower bounds of the latter two, denoted as ψ and ψ^*, are still functionals in bilinear form and are convex conjugate of each other and thus determines the limiting Wasserstein gradient flow with an effective Wasserstein distance $\overline{W}$; see the precise definition of these lower bounds in Theorem 4.1. The lower bound for ψ_ε^* is obtained by using a Fisher information reformulation in terms of $\sqrt{\frac{\rho^\varepsilon}{\pi_\varepsilon}}$ [2, 4] and a by now classical Γ-convergence technique for an associated Dirichlet energy. On the other hand, the lower bound for ψ_ε is obtained by a relaxation via the Legendre transformation and an upper bound estimate for ψ_ε^*. This requires one to overcome some regularity issues brought by the oscillations in the energy functional E_ε and the solution curve ρ^ε. This is achieved via a symmetric reformulation of the Fokker–Planck equation in terms of the variable $f^\varepsilon := \frac{\rho^\varepsilon}{\pi_\varepsilon}$.

We briefly mention some related references on Wasserstein gradient flow with multi-scale behaviours. Modelling of Fokker–Planck equation as a gradient flow in Wasserstein space was first noted by Jordan–Kinderlehrer–Otto [31]. They also show the convergence of a variational backward Euler scheme. There are many other evolutionary problems that can be formulated using multi-scale Wasserstein gradient flows; see for instance the porous medium equation [40] and more general aggregation-diffusion equations reviewed in ref. [14]. In [4], they use the evolutionary convergence of Wasserstein gradient flow to analyse the mean field equation in a zero noise limit for a reversible drift-diffusion process. There are also extensions for the zero noise limit from diffusion processes to chemical reactions described by time-changed Poisson processes on countable states; see [37] for the reversible case using a discrete Wasserstein gradient-flow approach and [24] for the irreversible case using a nonlinear semigroup approach for Hamilton–Jacobi equations. Homogenisation of action functionals on the space of probability measures has also been studied in [27]. In addition, convergence of Wasserstein gradient flows has been applied to related questions, which explore the mean-field limit and large deviation principle of weakly interacting particles; cf. [19, 9] and some recent developments in refs. [15, 17]. Furthermore, a similar convergence approach has also been used for generalised gradient flows and optimal transport on graphs and their diffusive limits. In various discrete settings, we refer to [26] for Gromov–Hausdorff convergence of discrete Wasserstein metrics, [20] for evolutionary Γ-convergence of finite volume scheme for linear Fokker–Planck equation [22, 23], for the homogenisation of Wasserstein distance on periodic graphs, and the recent works [50, 30, 28] for diffusive limits of some generalised gradient flows on graph.

The remainder of this paper is outlined as follows. In Section 2, we introduce the inhomogeneous Fokker–Planck and the ε-Wasserstein gradient flow in EDI form and describe our assumptions and main results. In Section 3, we obtain some uniform regularity estimates and convergence results for the ε-Wasserstein gradient flow. In Section 4, we pass the limit in the EDI form of the ε-Wasserstein gradient flow by proving lower bounds for the free energy and two dissipation functionals; see Theorem 4.1. In Section 5, we study the limiting gradient flow with respect to the induced limiting Wasserstein metric and compare it with the usual Gromov–Hausdorff convergence of W_ε.

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2. ε-system: inhomogeneous Fokker–Planck and generalised gradient flow

In this section, we introduce a spatially inhomogeneous Fokker–Planck equation, which, with fixed $\varepsilon > 0$, can be recast as a generalised gradient flow in ε-Wasserstein space in terms of a total energy given by a relative entropy. This Fokker–Planck equation is motivated by a drift-diffusion process with inhomogeneous noise and drift that satisfy the fluctuation–dissipation relation. In Section 2.3, we choose a pair of quadratic dissipation functionals $(\psi_\varepsilon, \psi_\varepsilon^*)$ which are convex conjugate to each other to recast the ε-Fokker–Planck equation as a generalised gradient flow in an EDI form. Then in Section 2.4, we state and explain our main results on the convergence of the gradient-flow structure as $\varepsilon \to 0$ and the resulting homogenised gradient flow of an effective free energy $\overline{E}$ with respect to an effective Wasserstein metric $\overline{W}$.

From now on, to avoid boundary effects, we work on periodic domain, denoted as $\Omega := \mathbb{T}^n$. Given any smooth potential function $U_\varepsilon : \Omega \longrightarrow \mathbb{R}$, consider the following (free) energy functional on $\mathcal{P}(\Omega)$

$$E_\varepsilon(\rho) = \int_\Omega U_\varepsilon(x)\rho(x)\,\mathrm{d}x + \int_\Omega \rho(x)\log\rho(x)\,\mathrm{d}x. \tag{2.1}$$

Let

$$\pi_\varepsilon(x) = e^{-U_\varepsilon(x)}. \tag{2.2}$$

Then (2.1) can be written in the form (1.12). The first variation $\dfrac{\delta E_\varepsilon}{\delta\rho}$ of E_ε is then given by

$$\frac{\delta E_\varepsilon}{\delta\rho}(\rho) = \log\rho + 1 + U_\varepsilon = \log\frac{\rho}{\pi_\varepsilon} + 1. \tag{2.3}$$

With a positive definite matrix B_ε, we consider the following inhomogeneous Fokker–Planck equation

$$\partial_t\rho_t^\varepsilon = \nabla\cdot\left(\rho_t^\varepsilon B_\varepsilon^{-1}\nabla\frac{\delta E_\varepsilon}{\delta\rho}(\rho_t^\varepsilon)\right) = \nabla\cdot\left(B_\varepsilon^{-1}\nabla\rho_t^\varepsilon + \rho_t^\varepsilon B_\varepsilon^{-1}\nabla U_\varepsilon\right). \tag{2.4}$$

The above equation can be interpreted in two ways. One is to regard it as the Kolmogorov forward equation of a drift-diffusion process with a multiplicative noise, while another as a gradient flow in a Wasserstein space $(\mathcal{P}(\Omega), W_\varepsilon)$ with the cost function defined in (1.3). We describe both of these in the following.

2.1. *ε-Fokker–Planck equation (2.4) as a Kolmogorov equation*

Consider a drift-diffusion process $(X_t)_{t\geq 0}$, described by the following stochastic differential equation

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t) * \mathrm{d}B_t, \tag{2.5}$$

where B_t is a one-dimensional Brownian motion, and

$$b(x) = -B_\varepsilon^{-1}(x)\nabla U_\varepsilon(x), \quad \text{and} \quad \sigma(x) = \sqrt{2B_\varepsilon^{-1}(x)}. \tag{2.6}$$

Here the multiplicative noise $\sigma(X_t) * \mathrm{d}B_t$ is in the backward Ito differential sense, which is equivalent to the forward Ito differential by adding an additional drift term

$$\sigma(X_t) * \mathrm{d}B_t = \frac{1}{2}\nabla\cdot(\sigma\sigma^T)(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t.$$

By Ito's formula, the generator of the process $(X_t)_{t\geq 0}$ is derived as follows. For any test function $\varphi \in C_b^2(\mathbb{R}^n)$ and initial condition $X_0 = x$, we compute

$$\lim_{t\to 0^+} \frac{\mathbb{E}^x[\varphi(X_t)] - \varphi(x)}{t} = \lim_{t\to 0^+} \mathbb{E}^x \frac{1}{t} \int_0^t \big[\nabla\varphi(X_s)\cdot b(X_s) + \frac{1}{2}\nabla^2\varphi(X_s) : (\sigma\sigma^T)(X_s) + \frac{1}{2}(\nabla\cdot(\sigma\sigma^T)(X_s))\cdot\nabla\varphi(X_s)\big]\,\mathrm{d}s \\ = \nabla\varphi(x)\cdot b(x) + \frac{1}{2}\nabla\cdot(\sigma\sigma^T\nabla\varphi(x)) =: \mathcal{L}\varphi. \tag{2.7}$$

Thus the corresponding Fokker–Planck equation to (2.5) is given by

$$\begin{aligned} \partial_t\rho_t^\varepsilon &= \mathcal{L}^*\rho_t^\varepsilon \\ &:= \frac{1}{2}\nabla\cdot\left(\sigma\sigma^T\nabla\rho_t^\varepsilon\right) - \nabla\cdot\left(\rho_t^\varepsilon b\right) \\ &= \nabla\cdot\left(B_\varepsilon^{-1}(x)\nabla\rho_t^\varepsilon(x)\right) + \nabla\cdot(\rho_t^\varepsilon(x)B_\varepsilon^{-1}(x)\nabla U_\varepsilon(x)), \end{aligned} \tag{2.8}$$

which is exactly (2.4). Note that the π_ε defined in (2.2), which is in the form of a Gibbs measure, is in fact the unique stationary distribution of (2.8), $\mathcal{L}^*\pi_\varepsilon = 0$.

We remark that in the above drift-diffusion process, we used the Ito backward differential to ensure that our process $(X_t)_{t\geq 0}$ with a multiplicative noise is reversible so that one can have a gradient flow structure for the corresponding Fokker–Planck equation. More precisely, we have that the diffusion process $(X_t)_{t\geq 0}$ (2.5) starting from $X_0 \sim \pi_\varepsilon$ is reversible in the sense that the time reversed process has the same distribution, i.e.

$$\mathbb{E}(\varphi_1(X_t)\varphi_2(X_0)|X_0\sim\pi_\varepsilon) = \mathbb{E}(\varphi_1(X_0)\varphi_2(X_t)|X_0\sim\pi_\varepsilon), \quad \forall\varphi_1,\varphi_2\in C_0^\infty(\mathbb{R}^n),\ \forall t>0. \tag{2.9}$$

This condition is equivalent to the symmetry of the generator $\mathcal{L}$ in $L^2(\pi_\varepsilon)$; cf. [25].

2.2. *ε -Fokker-Planck equation (2.4) as a gradient flow in* $(\mathcal{P}(\Omega), W_\varepsilon)$

Following Otto's formal Riemannian calculus on Wasserstein space [40], we now interpret the Fokker–Planck equation as a (negative) gradient flow in $(\mathcal{P}(\Omega), W_\varepsilon)$. For this purpose, we need to compute the Wasserstein gradient $\nabla^{W_\varepsilon}E_\varepsilon$ of E_ε in $(\mathcal{P}(\Omega), W_\varepsilon)$.

Given any absolutely continuous curve $\tilde{\rho}_t$ in $(\mathcal{P}(\Omega), W_\varepsilon)$ given by $\tilde{\rho}_t := (\chi_t)_\#\rho$ with $\tilde{\rho}_{t=0} = \rho$, where χ_t is the flow map induced by a smooth velocity field v_t. Then $\tilde{\rho}_t$ satisfies the continuity equation

$$\partial_t\tilde{\rho}_t + \nabla\cdot(\tilde{\rho}_t v_t) = 0.$$

With this, we compute the first variation of E_ε

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} E_\varepsilon(\tilde{\rho}_t) = \int_\Omega \frac{\delta E_\varepsilon}{\delta\rho}\partial_t\tilde{\rho}_t\Big|_{t=0}\mathrm{d}x = \int_\Omega \frac{\delta E_\varepsilon}{\delta\rho}\left(-\nabla\cdot(\tilde{\rho}_t v_t)\big|_{t=0}\right)\mathrm{d}x = \int_\Omega \left\langle\nabla\frac{\delta E_\varepsilon}{\delta\rho}, v_0\right\rangle\rho\,\mathrm{d}x. \tag{2.10}$$

We will use the above to identify the gradient $\nabla^{W_\varepsilon}E_\varepsilon$ of E_ε with respect to a Riemannian metric $\langle\cdot,\cdot\rangle_{T_\mathcal{P},T_\mathcal{P}}$ on the tangent plane $T_\mathcal{P}$ of $(\mathcal{P}(\Omega), W_\varepsilon)$.

Based on (1.5), we have that for any $\rho\in\mathcal{P}(\Omega)$ and $s_1, s_2 \in T_\mathcal{P}$ at ρ, the metric is given by

$$\langle s_1, s_2\rangle_{T_\mathcal{P},T_\mathcal{P}} := \int \rho(x)\left\langle B_\varepsilon^{-1}(x)\nabla\varphi_1(x), \nabla\varphi_2(x)\right\rangle dx, \quad \text{where } s_i = -\nabla\cdot(\rho B_\varepsilon^{-1}\nabla\varphi_i),\quad i=1,2. \tag{2.11}$$

A word is in place here to explain going from v_t in (1.5) to $\nabla\varphi$ above. At a fixed t and ρ_t, upon minimising $\int_\Omega \rho_t\langle B_\varepsilon(x)v_t, v_t\rangle\,\mathrm{d}x$ over v_t subject to $-\nabla\cdot(\rho_t v_t) = s\left(:=\frac{\partial\rho_t}{\partial t}\right)$, we have that

$$\int_\Omega \rho_t\langle B_\varepsilon(x)v_t, \xi\rangle\,\mathrm{d}x = 0 \ \text{ for all smooth vector field } \xi \text{ satisfying } -\nabla\cdot(\rho_t\xi) = 0.$$

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Hence, $B_\varepsilon v_t$ is orthogonal to all divergence free vector field of the form $\rho_t\xi$. We then conclude that $B_\varepsilon v_t$ must be the gradient of some (potential) function φ. Thus, v_t can be represented as $v_t = B_\varepsilon^{-1}\nabla\varphi$.

With the above, we express the first variation of E_ε using $\nabla^{W_\varepsilon}E_\varepsilon$ as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} E_\varepsilon(\tilde\rho_t) = \left\langle \nabla^{W_\varepsilon}E_\varepsilon, \partial_t\tilde\rho_t\big|_{t=0}\right\rangle_{T_\mathcal{P},T_\mathcal{P}} = \int_\Omega \rho\langle B_\varepsilon^{-1}\nabla\tilde\varphi, \nabla\varphi_0\rangle\,\mathrm{d}x, \tag{2.12}$$

where

$$\partial_t\tilde\rho_t\big|_{t=0} = -\nabla\cdot\left(\rho B_\varepsilon^{-1}\nabla\varphi_0\right) \quad \text{and} \quad \nabla^{W_\varepsilon}E_\varepsilon(\rho) = -\nabla\cdot\left(\rho B_\varepsilon^{-1}\nabla\tilde\varphi\right). \tag{2.13}$$

Comparing (2.10) with (2.12), we have

$$\int_\Omega \left\langle \nabla\frac{\delta E_\varepsilon}{\delta\rho}, v_0\right\rangle \rho\,\mathrm{d}x = \int_\Omega \rho\big\langle B_\varepsilon^{-1}\nabla\tilde\varphi, \nabla\varphi_0\big\rangle\,\mathrm{d}x$$

which is set to hold for any $v_0 = B_\varepsilon^{-1}\nabla\varphi_0$. Hence, $\nabla\tilde\varphi = \nabla\dfrac{\delta E_\varepsilon}{\delta\rho}$. Thus, the second part of (2.13) leads to the following identification of $\nabla^{W_\varepsilon}E(\rho)$,

$$\nabla^{W_\varepsilon}E_\varepsilon(\rho) := -\nabla\cdot\left(\rho B_\varepsilon^{-1}\nabla\frac{\delta E_\varepsilon}{\delta\rho}\right) = -\nabla\cdot\left(\rho B_\varepsilon^{-1}\nabla\log\frac{\rho}{\pi_\varepsilon}\right). \tag{2.14}$$

Hence, the inhomogeneous Fokker–Planck equation (2.4) indeed can be written as a gradient flow of E_ε with respect to the ε-Wasserstein metric W_ε, i.e.,

$$\partial_t\rho_t^\varepsilon = -\nabla^{W_\varepsilon}E_\varepsilon(\rho_t^\varepsilon) = \nabla\cdot\left(\rho^\varepsilon B_\varepsilon^{-1}\nabla\log\frac{\rho^\varepsilon}{\pi_\varepsilon}\right). \tag{2.15}$$

We remark that in general an equation may have many different gradient flow structures with respect to the same free energy E_ε, cf. [38]. However, in this paper, we restrict ourselves within the framework of Wasserstein gradient flows as it fits naturally to the evolution in probability space.

2.3. *ε -generalised gradient flow in energy-dissipation inequality (EDI) form*

As mentioned previously, in order to study the limiting gradient flow structure as the small parameter $\varepsilon\to 0$ in our ε-gradient flow (2.15), we will recast it in an energy-dissipation inequality (EDI) form (1.13) that is shown to be equivalent to the original ε-gradient flow system.

Denote the ε-dissipation on the tangent plane $T_\mathcal{P}$ as a functional $\psi_\varepsilon : \mathcal{P}\times T_\mathcal{P}\to\mathbb{R}$ defined by

$$\psi_\varepsilon(\rho, s) := \frac{1}{2}\int_\Omega \langle\nabla u, B_\varepsilon^{-1}\nabla u\rangle\rho\,\mathrm{d}x, \qquad \text{with } s = -\nabla\cdot\left(\rho B_\varepsilon^{-1}\nabla u\right), \tag{2.16}$$

and the ε-dissipation on the cotangent plane $T_\mathcal{P}^*$ as a functional $\psi_\varepsilon^* : \mathcal{P}\times T_\mathcal{P}^*\to\mathbb{R}$ defined by

$$\psi_\varepsilon^*(\rho,\xi) := \frac{1}{2}\int_\Omega \langle\nabla\xi, B_\varepsilon^{-1}\nabla\xi\rangle\rho\,\mathrm{d}x. \tag{2.17}$$

It is easy to check that

$$\begin{aligned}\psi_\varepsilon(\rho, s) &= \sup_{\xi\in T_\rho^*}\left\{\langle\xi, s\rangle_{T_\rho^*,T_\rho} - \psi_\varepsilon^*(\rho,\xi)\right\}\\ &= \langle\xi^*, s\rangle_{T_\rho^*,T_\rho} - \psi_\varepsilon^*(\rho,\xi^*) \qquad \text{with } s = -\nabla\cdot\left(\rho B_\varepsilon^{-1}\nabla\xi^*\right)\\ &= \frac{1}{2}\int_\Omega\langle\nabla\xi^*, B_\varepsilon^{-1}\nabla\xi^*\rangle\rho\,\mathrm{d}x.\end{aligned} \tag{2.18}$$

Applying the Fenchel–Young inequality to the convex functionals ψ_ε and ψ_ε^*, we have

$$\langle\xi, s\rangle \le \psi_\varepsilon^*(\rho,\xi) + \psi_\varepsilon(\rho, s), \qquad \text{for all} \quad \xi\in T_\rho^*, \quad \text{and} \quad s\in T_\rho, \tag{2.19}$$

with equality holds if and only if $\xi\in\partial_s\psi_\varepsilon(\rho, s)$ and $s\in\partial_\xi\psi_\varepsilon^*(\rho,\xi)$. Here $\partial_s\psi_\varepsilon(\rho, s)$ and $\partial_\xi\psi_\varepsilon^*(\rho,\xi)$ refer to the subdifferentials of ψ_ε and ψ_ε^* on T_ρ and T_ρ^*, respectively, at a fixed ρ. We also note the following.

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(1) For all $\eta \in T^*_{\mathcal{P}}$, we have

$$\left\langle \partial_\xi \psi^*_\varepsilon(\rho,\xi),\eta\right\rangle = \lim_{\tau\to 0}\frac{d}{d\tau}\psi^*_\varepsilon(\rho,\xi+\tau\eta)\Big|_{\tau=0}$$
$$= \int \langle \nabla\xi, B_\varepsilon^{-1}\nabla\eta\rangle\rho \ \mathrm{d}x = \int -\eta\nabla\cdot(\rho B_\varepsilon^{-1}\nabla\xi)\ \mathrm{d}x$$

so that $\partial_\xi\psi^*_\varepsilon(\rho,\xi) = -\nabla\cdot(\rho B_\varepsilon^{-1}\nabla\xi)$. Hence, $s\in\partial_\xi\psi^*_\varepsilon(\rho,\xi)$ means $s = -\nabla\cdot(\rho B_\varepsilon^{-1}\nabla\xi)$.

(2) For all $\sigma\in T_{\mathcal{P}}$, we have

$$\left\langle \partial_s \psi_\varepsilon(\rho,s),\sigma\right\rangle = \lim_{\tau\to 0}\frac{d}{d\tau}\psi_\varepsilon(\rho,s+\tau\sigma)\Big|_{\tau=0}$$
$$= \int \langle \nabla u, B_\varepsilon^{-1}\nabla\omega\rangle\rho \ \mathrm{d}x = \int -u\nabla\cdot(\rho B_\varepsilon^{-1}\nabla\omega)\ \mathrm{d}x$$
$$\left(\text{where}\quad s = -\nabla\cdot(\rho B_\varepsilon^{-1}\nabla u),\quad \sigma = -\nabla\cdot(\rho B_\varepsilon^{-1}\nabla\omega)\right)$$
$$= \int u\sigma\ \mathrm{d}x$$

so that $\partial_s\psi_\varepsilon(\rho,s) = u$. Hence, $\xi\in\partial_s\psi_\varepsilon(\rho,s)$ means ξ satisfies $s = -\nabla\cdot(\rho B_\varepsilon^{-1}\nabla\xi)$.

With the above, we now reformulate (2.15) in the form of an EDI. To this end, we compute,

$$\frac{d}{dt}E_\varepsilon(\rho^\varepsilon_t) = \left\langle \frac{\delta E_\varepsilon}{\delta\rho}, \partial_t\rho^\varepsilon_t\right\rangle,\quad\text{or}\quad \frac{d}{dt}E_\varepsilon(\rho^\varepsilon_t) + \left\langle -\frac{\delta E_\varepsilon}{\delta\rho}, \partial_t\rho^\varepsilon_t\right\rangle = 0. \tag{2.20}$$

By (2.19), $\partial_t\rho^\varepsilon_t = -\nabla^{W_\varepsilon}E_\varepsilon(\rho^\varepsilon_t) = \nabla\cdot\left(\rho B_\varepsilon^{-1}\nabla\frac{\delta E_\varepsilon}{\delta\rho}\right)$ if and only if

$$\psi_\varepsilon(\rho^\varepsilon_\tau,\partial_\tau\rho^\varepsilon_\tau) + \psi^*_\varepsilon\left(\rho^\varepsilon_\tau, -\frac{\delta E_\varepsilon}{\delta\rho}(\rho^\varepsilon_\tau)\right) \le \left\langle -\frac{\delta E_\varepsilon}{\delta\rho}, \partial_t\rho^\varepsilon_t\right\rangle.$$

Hence, upon integrating (2.20), our gradient flow (2.15) is equivalent to the following:

$$E_\varepsilon(\rho^\varepsilon_t) + \int_0^t\left[\psi_\varepsilon(\rho^\varepsilon_\tau,\partial_\tau\rho^\varepsilon_\tau) + \psi^*_\varepsilon\left(\rho^\varepsilon_\tau, -\frac{\delta E_\varepsilon}{\delta\rho}(\rho^\varepsilon_\tau)\right)\right]\mathrm{d}\tau \le E_\varepsilon(\rho^\varepsilon_0). \tag{2.21}$$

We note that the very first step, (2.20) is a crucial chain rule of differentiation. This is justified in our paper due to the regularity property of our energy functional and the solution. Precise statements will be given in Section 3. In general (for example, discrete or general metric space) settings, the absolute continuity of $E_\varepsilon(\rho^\varepsilon_t)$ (in time) and the validity of the chain rule (2.20) need to be proved; cf., [29, 28].

Before leaving this section, for convenience, we write down the following explicit expressions.

$$\psi^*_\varepsilon\left(\rho^\varepsilon_\tau, -\frac{\delta E_\varepsilon}{\delta\rho}(\rho^\varepsilon_\tau)\right) = \int_\Omega\left\langle\nabla\left(\frac{\delta E_\varepsilon}{\delta\rho}\right), B_\varepsilon^{-1}\nabla\left(\frac{\delta E_\varepsilon}{\delta\rho}\right)\right\rangle\rho^\varepsilon_t\ \mathrm{d}x$$
$$= \frac{1}{2}\int_\Omega\left\langle\nabla\log\frac{\rho^\varepsilon_\tau}{\pi_\varepsilon}, B_\varepsilon^{-1}\nabla\log\frac{\rho^\varepsilon_\tau}{\pi_\varepsilon}\right\rangle\rho^\varepsilon_\tau\,\mathrm{d}x, \tag{2.22}$$

and

$$\psi_\varepsilon\left(\rho^\varepsilon_\tau,\partial_\tau\rho^\varepsilon_\tau\right) = \frac{1}{2}\int_\Omega\langle\nabla u, B_\varepsilon^{-1}\nabla u\rangle\rho^\varepsilon_\tau\,\mathrm{d}x,\qquad\text{with}\quad -\nabla\cdot\left(\rho^\varepsilon_\tau B_\varepsilon^{-1}\nabla u\right) = \partial_\tau\rho^\varepsilon_\tau. \tag{2.23}$$

2.4. Main results

Briefly stated, our main result is that the gradient-flow structure is preserved in the limit, i.e., (2.15) converges to a limiting gradient flow. More precisely, the solution ρ^ε_t of (2.15) converges (weakly) to ρ_t

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that solves a gradient flow with respect to a limiting Wasserstein distance $\overline{W}$,

$$\partial_t \rho_t = -\nabla^{\overline{W}} \overline{E}(\rho_t) = \nabla \cdot \left(\rho_t \overline{B}^{-1} \nabla \log \frac{\rho_t}{\overline{\pi}} \right). \tag{2.24}$$

In the above, the limiting energy is given as

$$\overline{E}(\rho) = \mathrm{KL}(\rho || \overline{\pi}) = \int_\Omega \rho \log \frac{\rho}{\overline{\pi}} \, \mathrm{d}x, \tag{2.25}$$

where the $\overline{\pi}$ is simply the spatial average of π_ε with respect to some fast variable – see (2.33) below. The matrix $\overline{B}$ is obtained by taking appropriate average of B_ε over the fast variable weighted by the solution of a cell problem (A.9) or equivalently, by considering the Γ-limit of a variational functional (Theorem 4.2). The Wasserstein distance $\overline{W}$ is related to $\overline{B}$ just as the way W_ε is related to B_ε – see Section 5.1.

Similar to (2.21), (2.16), and (2.17), equation (2.24) is formulated as an EDI, i.e.,

$$\overline{E}(\rho_t) + \int_0^t \left[\psi(\rho_\tau, \partial_\tau \rho_\tau) + \psi^* \left(\rho_\tau, -\frac{\delta \overline{E}}{\delta \rho}(\rho_\tau) \right) \right] \mathrm{d}\tau \le \overline{E}(\rho_0), \tag{2.26}$$

where $\psi^* : \mathcal{P} \times T^*_{\mathcal{P}} \to \mathbb{R}$ is the limiting dissipation functional on the cotangent plane $T^*_{\mathcal{P}}$ given by

$$\psi^*(\rho, \xi) := \frac{1}{2} \int_\Omega \langle \nabla \xi, \bar{B}^{-1} \nabla \xi \rangle \rho \, \mathrm{d}x, \tag{2.27}$$

and $\psi : \mathcal{P} \times T_{\mathcal{P}} \to \mathbb{R}$ is the limiting dissipation functional on the tangent plane $T_{\mathcal{P}}$ given by

$$\psi(\rho, s) := \frac{1}{2} \int_\Omega \langle \nabla u, \bar{B}^{-1} \nabla u \rangle \rho \, \mathrm{d}x, \qquad \text{with } s = -\nabla \cdot \left(\rho \bar{B}^{-1} \nabla u \right). \tag{2.28}$$

The precise statement of the convergence of (2.21) to (2.26) will be given in Section 4, Theorem 4.1.

Curiously, under the current setting, $\overline{W}$ is *not* the Gromov–Hausdorff limit W_{GH} of W_ε which is the common mode of convergence for metric spaces, cf. [53, 26, 21]. In Section 5.2, We have constructed examples such that $\overline{W}$ is *strictly bigger* than W_{GH}. We believe that this statement is true for general heterogeneous media.

Before proceeding further, we introduce the following notations and conventions. As we will often consider functions that oscillate on a small length scale, $0 < \varepsilon \ll 1$, it is convenient to introduce the following fast variable

$$y = \frac{x}{\varepsilon}. \tag{2.29}$$

The domain for y is taken to be the n-dimensional torus $\mathbb{T}^n$ when the oscillatory functions are 1-periodic in y. The notation $\overline{A}$ means that it is derived from some averaging of A over the fast variable y. For time-dependent problems, we often deal with functions defined on both space and time variables x, t. For ease of notation, given a function $f = f(x, t)$, we often use f_t to denote $f_t(\,\cdot\,)$, i.e., the slice of f at a fixed time t. We will use $\rightharpoonup$ and $\longrightarrow$ to denote weak and strong convergence in some function spaces. Two common spaces used are the space of probability measures $\mathcal{P}(\Omega)$ and $L^p(\Omega)$ spaces. The value of p will depend on contexts. For the convergence of a sequence of functions f_ε as $\varepsilon \to 0$, we will use the same notation even if the convergence only holds upon extraction of subsequence. (The convergence can be established for the whole sequence if the limiting equation has unique solution which is the case for our linear Fokker–Planck equation (2.24).)

Next we state the main assumptions for our results. Some of these are made only for simplicity. They can be relaxed if we choose to use more technical tools.

(i) Recall that the domain Ω is taken to be an n-dimensional torus $\mathbb{T}^n$. This is not to be confused with the $\mathbb{T}^n$ for the fast variable y. We note that the boundedness of the domain can be removed, allowing one to work in $P_2(\mathbb{R}^n)$ if a confinement potential U is incorporated in the dynamics. Other boundary conditions, such as Dirichlet or no-flux conditions, may also be considered.

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(ii) For B_ε, we consider

$$B_\varepsilon(x) = B\left(\frac{x}{\varepsilon}\right), \quad \text{or} \quad B_\varepsilon(x) = B(y), \tag{2.30}$$

where $B(\cdot)$ is 1-periodic. Furthermore, $B(\cdot)$ is bounded and uniformly positive definite, i.e., there are $C_1, C_2 > 0$ such that for all $y \in \mathbb{T}^n$, it holds that

$$C_1 I \leq B(y) \leq C_2 I. \tag{2.31}$$

This form of B_ε can certainly be generalised to allow for dependence on the slow variable: $B_\varepsilon(x) = B(x, \frac{x}{\varepsilon})$. For simplicity, we assume further that B is smooth in y.

(iii) For π_ε, we consider the following form of separation of length scales:

$$\pi_\varepsilon(x) = \pi\left(x, \frac{x}{\varepsilon}\right). \tag{2.32}$$

In the above, π is 1-periodic in the fast variable $y = \dfrac{x}{\varepsilon}$. We further assume that π is smooth in both x and y and is bounded away from zero and from above uniformly in $\varepsilon > 0$. The following notation referring to an averaged version of π will be used in this paper:

$$\overline{\pi}(x) = \int \pi(x, y)\, \mathrm{d}y. \tag{2.33}$$

As concrete examples, π_ε can be taken as

$$\pi_\varepsilon^{\mathrm{I}}(x) = \pi_0(x) + \pi_1\left(x, \frac{x}{\varepsilon}\right), \quad \text{or} \quad \pi_\varepsilon^{\mathrm{II}}(x) = \pi_0(x) + \varepsilon\pi_1\left(x, \frac{x}{\varepsilon}\right). \tag{2.34}$$

Then $\pi_\varepsilon^{\mathrm{I}}$ and $\pi_\varepsilon^{\mathrm{II}}$ converge as follow:

$$\pi_\varepsilon^{\mathrm{I}}(x) \rightharpoonup \overline{\pi}^{\mathrm{I}}(x) := \pi_0(x) + \int_{\mathbb{T}^n} \pi_1(x, y)\ \mathrm{d}y, \quad \text{and} \quad \pi_\varepsilon^{\mathrm{II}}(x) \longrightarrow \overline{\pi}^{\mathrm{II}}(x) := \pi_0(x). \tag{2.35}$$

We thus call $\pi_\varepsilon^{\mathrm{I}}$ the oscillatory case while $\pi_\varepsilon^{\mathrm{II}}$ the uniform case. (We refer to the work [19] for large deviations for multiscale diffusion with $\pi_\varepsilon^{\mathrm{II}}$.)

(iv) The initial data ρ_0^ε is bounded away from zero and from above uniformly in $\varepsilon > 0$. It is assumed to be well-prepared in the following sense,

$$\text{there is a } \rho_0 \text{ such that as } \varepsilon \to 0, \text{ it holds } \rho_0^\varepsilon \rightharpoonup \rho_0 \text{ and } E_\varepsilon(\rho_0^\varepsilon) \to \overline{E}(\rho_0), \quad \text{as } \varepsilon \to 0, \tag{2.36}$$

where E_ε and $\overline{E}$ are given by (2.1) and (2.25). More precise smoothness requirements on ρ_0 will be listed in Lemmas 3.1, 3.3, and Corollaries 3.2 and 3.4.

We have the following remarks about our results.

Remark 2.1.

(1) *As $\pi_\varepsilon^{\mathrm{II}}$ can be treated as a special case of $\pi_\varepsilon^{\mathrm{I}}$, or more generally, of π_ε, we will concentrate on the proof for π_ε. Our result is also consistent with the statement obtained by using the asymptotic expansion described in Appendix A. At the end of that section, we also make some remarks about the revised statement for $\pi_\varepsilon^{\mathrm{II}}$.*

(2) *The approach we take resembles the work of Forkert–Maas–Portinale [20] on the convergence of a finite volume scheme for a Fokker–Planck equation. By and large, the framework of their (numerical) approximation enjoys stronger regularity, while our current problem concentrates on the oscillation of the underlying medium.*

3. Some a-priori estimates

In order to study the asymptotic behaviour as $\varepsilon \to 0$, we first establish some a-priori estimates for our ε-gradient flow system (2.4) (or (2.15)). These would then give us the space-time compactness and

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convergence. These variational estimates for linear parabolic equations are standard but we give a brief proof for completeness.

First, we recast (2.4) as

$$\partial_t \rho_t^\varepsilon = \nabla \cdot \left(\pi_\varepsilon B_\varepsilon^{-1} \nabla \frac{\rho_t^\varepsilon}{\pi_\varepsilon} \right). \tag{3.1}$$

Denote $f_t^\varepsilon := \dfrac{\rho_t^\varepsilon}{\pi_\varepsilon}$. Then f_t^ε satisfies the following *backward* equation

$$\partial_t f_t^\varepsilon = \frac{1}{\pi_\varepsilon} \nabla \cdot \left(\pi_\varepsilon B_\varepsilon^{-1} \nabla f_t^\varepsilon \right) =: L_\varepsilon(f_t^\varepsilon). \tag{3.2}$$

It is easy to verify that L_ε is self-adjoint in $L^2(\pi_\varepsilon)$, i.e.,

$$\langle L_\varepsilon u, v\rangle_{\pi_\varepsilon} = \langle u, L_\varepsilon v\rangle_{\pi_\varepsilon}, \quad \forall u, v \in L^2(\pi_\varepsilon), \tag{3.3}$$

where $\langle \cdot, \cdot \rangle_{\pi_\varepsilon}$ denotes the π_ε-weighted L^2 inner product, $\langle u, v\rangle_{\pi_\varepsilon} := \int_\Omega u(x)v(x)\pi_\varepsilon(x)\,\mathrm{d}x$.

We recall here the standing assumptions of uniform positive definiteness of B_ε and uniform positivity and boundedness of π_ε as stated in (2.30) and (2.32) in Section 2.4. We then have the following uniform estimates for f_t^ε.

Lemma 3.1. *Let f_0^ε be the initial data for (3.2). We define,*

$$A_0 := \sup_{\varepsilon>0} \int_\Omega (f_0^\varepsilon)^2 \pi_\varepsilon \,\mathrm{d}x, \tag{3.4}$$

$$B_0 := \sup_{\varepsilon>0} \int_\Omega \langle \nabla f_0^\varepsilon, B_\varepsilon^{-1}\pi_\varepsilon \nabla f_0^\varepsilon \rangle \,\mathrm{d}x. \tag{3.5}$$

Let $0 < T < \infty$ be given. We have the following statements.

(1) If $0 < m_0 < \inf f_0^\varepsilon < M_0 < \infty$ on Ω for some finite positive constants m_0 and M_0, then $m_0 < \inf f_t^\varepsilon < M_0$ for all $t > 0$.

(2) If $A_0 < \infty$, then $f^\varepsilon \in L^\infty((0,T);L^2(\Omega)) \bigcap L^2((0,T);H^1(\Omega))$ with the following uniform-in-ε bound: for all $0 < t < T$,

$$\frac{1}{2}||f_t^\varepsilon||_{\pi_\varepsilon}^2 + \int_0^t \int_\Omega \langle \nabla f_s^\varepsilon, B_\varepsilon^{-1}\pi_\varepsilon \nabla f_s^\varepsilon\rangle \,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}||f_0^\varepsilon||_{\pi_\varepsilon}^2 \le A_0. \tag{3.6}$$

(3) If $B_0 < \infty$ (which by Poincare inequality implies $A_0 < \infty$), then

$$f^\varepsilon \in L^\infty((0,T); H^1(\Omega)) \bigcap H^1((0,T); L^2(\Omega))$$

with the following uniform-in-ε bound: for all $0 < t < T$,

$$\frac{1}{2}\int_\Omega \langle \nabla f_0^\varepsilon, B_\varepsilon^{-1}\pi_\varepsilon \nabla f_0^\varepsilon\rangle \,\mathrm{d}x + \int_0^t \int_\Omega (\partial_s f_s^\varepsilon)^2 \pi_\varepsilon \,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}\int_\Omega \langle \nabla f_0^\varepsilon, B_\varepsilon^{-1}\pi_\varepsilon \nabla f_0^\varepsilon\rangle \,\mathrm{d}x \le \frac{B_0}{2}. \tag{3.7}$$

From (3.2) and $\int_0^t \int_\Omega (\partial_s f_s^\varepsilon)^2 \pi_\varepsilon \,\mathrm{d}x\,\mathrm{d}s \le \frac{B_0}{2}$, we also have

$$\sup_{\varepsilon>0} \int_0^T \int_\Omega \left(\nabla \cdot \left(B_\varepsilon^{-1}\pi_\varepsilon \nabla f_s^\varepsilon \right) \right)^2 \mathrm{d}x\,\mathrm{d}s < \infty. \tag{3.8}$$

Proof. Note that

$$\partial_t f_t^\varepsilon = B_\varepsilon^{-1} : D^2 f_t^\varepsilon + \frac{1}{\pi_\varepsilon}\left\langle \nabla(B_\varepsilon^{-1}\pi_\varepsilon), \nabla f_t^\varepsilon \right\rangle.$$

By the positive definitenss of B_ε, statement (1) then follows directly from maximum principle.

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Next, both (3.6) and (3.7) follows from simple energy identity. For the former, we compute

$$\frac{d}{dt}\frac{1}{2}||f_t^\varepsilon||^2_{\pi_\varepsilon} = \int_\Omega f_t^\varepsilon \partial f_t^\varepsilon \pi_\varepsilon \,\mathrm{d}x = -\int_\Omega \langle \nabla f_t^\varepsilon, B_\varepsilon^{-1}\pi_\varepsilon \nabla f_t^\varepsilon\rangle \,\mathrm{d}x.$$

Integration in time from 0 to t gives (3.6).

For (3.7), we compute

$$\frac{d}{dt}\frac{1}{2}\int_\Omega \langle \nabla f_t^\varepsilon, B_\varepsilon^{-1}\pi_\varepsilon \nabla f_t^\varepsilon\rangle \,\mathrm{d}x = \int_\Omega \langle \nabla \partial_t f_t^\varepsilon, B_\varepsilon^{-1}\pi_\varepsilon \nabla f_t^\varepsilon\rangle \,\mathrm{d}x$$
$$= -\int_\Omega \partial_t f_t^\varepsilon \nabla\cdot\left(B_\varepsilon^{-1}\pi_\varepsilon \nabla f_t^\varepsilon\right)\mathrm{d}x = -\int_\Omega (\partial_t f_t^\varepsilon)^2 \pi_\varepsilon \,\mathrm{d}x.$$

Integration in time from 0 to t again gives the result. Estimate (3.8) follows from definition. □

The above and Fubini's Theorem immediately leads to the following compactness results.

Corollary 3.2. *If $B_0 < \infty$, then there is a subsequence f^ε and an $f \in L^2(0,T;L^2(\Omega))$ such that $f^\varepsilon \longrightarrow f$ in $L^2(0,T;L^2(\Omega))$, i.e.,*

$$\int_0^T \int_\Omega |f_t^\varepsilon - f_t|^2 \,\mathrm{d}x\,\mathrm{d}t \to 0. \tag{3.9}$$

Furthermore, we have

$$\int_\Omega |f_t^\varepsilon - f_t|^2 \,\mathrm{d}x \to 0 \quad \textit{for a.e. } t\in[0,T]. \tag{3.10}$$

For our application, we will also need some regularity estimates for the time derivative of f^ε. Define $h_t^\varepsilon := \partial_t f_t^\varepsilon$. Then it satisfies the same equation (3.2), i.e.,

$$\partial_t h_t^\varepsilon = \frac{1}{\pi_\varepsilon}\nabla\cdot\left(\pi_\varepsilon B_\varepsilon^{-1}\nabla h_t^\varepsilon\right) =: L_\varepsilon(h_t^\varepsilon). \tag{3.11}$$

As a direct application of Lemma 3.1 and Corollary 3.2, we have the following lemma and corollary.

Lemma 3.3. *Let $h_0^\varepsilon = \partial_t f_t^\varepsilon|_{t=0}$ be the initial data for (3.11). We define,*

$$C_0 := \sup_{\varepsilon>0}\int_\Omega (h_0^\varepsilon)^2\pi_\varepsilon \,\mathrm{d}x \left(= \sup_{\varepsilon>0}\int_\Omega (\partial_t f_0^\varepsilon)^2\pi_\varepsilon \,\mathrm{d}x\right), \tag{3.12}$$

$$D_0 := \sup_{\varepsilon>0}\int_\Omega \langle \nabla h_0^\varepsilon, B_\varepsilon^{-1}\pi_\varepsilon \nabla h_0^\varepsilon\rangle \,\mathrm{d}x \left(= \sup_{\varepsilon>0}\int_\Omega \langle \nabla(\partial_t f_0^\varepsilon), B_\varepsilon^{-1}\pi_\varepsilon \nabla(\partial_t f_0^\varepsilon)\rangle \,\mathrm{d}x\right). \tag{3.13}$$

Let $0 < T < \infty$ be given. We have the following statements.

(1) If $C_0 < \infty$, then $h^\varepsilon \in L^\infty((0,T);L^2(\Omega)) \bigcap L^2((0,T);H^1(\Omega))$. In particular, for all $0<t<T$, we have the following identity,

$$\frac{1}{2}||h_t^\varepsilon||^2_{\pi_\varepsilon} + \int_0^t \int_\Omega \langle \nabla h_s^\varepsilon, B_\varepsilon^{-1}\pi_\varepsilon \nabla h_s^\varepsilon\rangle \,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}||h_0^\varepsilon||^2_{\pi_\varepsilon}. \tag{3.14}$$

(2) If $D_0 < \infty$, then $h^\varepsilon \in L^\infty((0,T);H^1(\Omega)) \bigcap H^1((0,T);L^2(\Omega))$. In particular, for all $0<t<T$, we have the following identity,

$$\frac{1}{2}\int_\Omega \langle \nabla h_t^\varepsilon, B_\varepsilon^{-1}\pi_\varepsilon \nabla h_t^\varepsilon\rangle \,\mathrm{d}x + \int_0^t\int_\Omega (\partial_s h_s^\varepsilon)^2\pi_\varepsilon \,\mathrm{d}x\,\mathrm{d}s = \frac{1}{2}\int_\Omega \langle \nabla h_0^\varepsilon, B_\varepsilon^{-1}\pi_\varepsilon \nabla h_0^\varepsilon\rangle \,\mathrm{d}x \tag{3.15}$$

Corollary 3.4. *If $D_0 < \infty$, then there is a subsequence h^ε and an $h \in L^2(0,T;L^2(\Omega)$ such that $h^\varepsilon \longrightarrow h$ in $L^2(0,T;L^2(\Omega))$, i.e.*

$$\int_0^T \int_\Omega |h_t^\varepsilon - h_t|^2 \,\mathrm{d}x\,\mathrm{d}t \to 0. \tag{3.16}$$

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Furthermore, we have

$$\int_{\Omega} |h_t^{\varepsilon} - h_t|^2 \,\mathrm{d}x \to 0, \quad \textit{for a.e. } t \in [0,T]. \tag{3.17}$$

Recall Assumption (iii) in Section 2.4 for the invariant measure π_ε. For the convenience of our upcoming proof, we collect the necessary convergence results in the following lemma.

Lemma 3.5. *Suppose A_0, B_0, C_0 and $D_0 < \infty$. Then (from Lemmas 3.1 and 3.3) we have*

$$f^{\varepsilon} \in L^{\infty}((0,T);H^1(\Omega)) \bigcap H^1((0,T);L^2(\Omega)), \quad \textit{and} \quad \partial_t f^{\varepsilon} \in L^{\infty}((0,T);H^1(\Omega)) \bigcap H^1((0,T);L^2(\Omega)). \tag{3.18}$$

Furthermore (from Corollaries 3.2 and 3.4), up to ε-subsequence, we have

$$f^{\varepsilon} \longrightarrow f, \quad \textit{and} \quad \partial_t f^{\varepsilon} \longrightarrow \partial_t f \quad \textit{in } L^2((0,T);L^2(\Omega)). \tag{3.19}$$

Upon defining $\rho_t = f_t \overline{\pi}$, we have

$$\frac{\rho^{\varepsilon}}{\pi_{\varepsilon}} \ (= f^{\varepsilon}) \longrightarrow \frac{\rho}{\overline{\pi}} \ (= f) \quad \textit{in } L^2((0,T);L^2(\Omega)), \tag{3.20}$$

and

$$\rho^{\varepsilon} \rightharpoonup \rho \quad \textit{in } L^2((0,T);L^2(\Omega)), \tag{3.21}$$

$$\frac{\partial_t \rho^{\varepsilon}}{\pi_{\varepsilon}} \ (= \partial_t f^{\varepsilon}) \longrightarrow \frac{\partial_t \rho}{\overline{\pi}} \ (= \partial_t f) \quad \textit{in } L^2((0,T);L^2(\Omega)), \tag{3.22}$$

$$\partial_t \rho^{\varepsilon} \rightharpoonup \partial_t \rho \quad \textit{in } L^2((0,T);L^2(\Omega)). \tag{3.23}$$

Instead of strong and weak convergence in $L^2(0,T;L^2(\Omega))$, by (3.10) and (3.17), statements (3.19) – (3.23) also hold with the same respective strong and weak topologies in $L^2(\Omega)$ for a.e. $t \in [0,T]$.

Remark 3.6. *Note that currently our approach does require a high degree of regularity for the initial data. Its existence and construction would require the characterisation of precise oscillations of the solution which in principle can be done by considering second and higher order cell problems. However, we believe this requirement can be much relaxed by means of parabolic regularity. For example, if $A_0 < \infty$, then $f_t^{\varepsilon} \in H^1(\Omega)$ for some $t > 0$ and if $B_0 < \infty$, then $\partial_t f_t^{\varepsilon} \in L^2(\Omega)$ for some $t > 0$. This can be iterated due to the variational structure of equation (3.2). Alternatively, we can opt to utilise some technical results similar to [31, p.14, steps (a – c)] and [20, Proposition 4.4] in which the initial data even belongs to $L^1(\Omega)$. For simplicity, in this paper, we do not pursue this route, as we consider it beyond the scope of homogenisation which is our key motivation.*

The final statement in this section gives the time continuity of ρ_t^{ε} in the standard Wasserstein space $(\mathcal{P}(\Omega), W_2)$ (1.8).

Lemma 3.7. *Assume $E_\varepsilon(\rho_0^{\varepsilon}) < +\infty$. For any $T > 0$, let $\rho_t^{\varepsilon}, t \in [0,T]$ be a solution to the ε-gradient flow system (2.21). Then there is $0 < C < \infty$ such that*

$$W_2^2(\rho_t^{\varepsilon}, \rho_s^{\varepsilon}) \le C|t-s|, \quad \forall\, 0 \le s \le t \le T, \tag{3.24}$$

where $W_2(\cdot,\cdot)$ is the standard W_2-distance. Consequently, there exist a subsequence ρ^{ε} and $\rho \in C([0,T];\mathcal{P}(\Omega))$ such that

$$W_2^2(\rho_t^{\varepsilon}, \rho_t) \to 0, \quad \textit{uniformly in } t \in [0,T]. \tag{3.25}$$

Proof. First, since $\rho_t^{\varepsilon}, t \in [0,T]$ satisfies (2.21) and $E_\varepsilon(\rho_0^{\varepsilon}) < +\infty$, we have for any $0 \le s \le t \le T$,

$$\int_s^t \psi_\varepsilon(\rho_\tau^{\varepsilon}, \partial_\tau \rho_\tau^{\varepsilon}) \,\mathrm{d}\tau < +\infty. \tag{3.26}$$

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This means for the curve $\rho_t^\varepsilon, t \in [0,T]$ with $\partial_t \rho_t^\varepsilon = -\nabla \cdot \left(\rho_t^\varepsilon B_\varepsilon^{-1} \nabla u_t^\varepsilon\right)$, we have

$$\int_s^t \int_\Omega \frac{1}{2} \langle \nabla u_\tau^\varepsilon, B_\varepsilon^{-1} \nabla u_\tau^\varepsilon \rangle \rho_\tau^\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau < +\infty. \tag{3.27}$$

For this curve, the velocity in the continuity equation is given by $v_t^\varepsilon = B_\varepsilon^{-1} \nabla u_t^\varepsilon$. From [1, Theorem 17.2], we have

$$\begin{aligned} W_2^2(\rho_t^\varepsilon, \rho_s^\varepsilon) \leq |t-s| \int_s^t \int_\Omega |v_\tau^\varepsilon|^2 \rho_\tau^\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau =& |t-s| \int_s^t \int_\Omega |B_\varepsilon^{-1} \nabla u_\tau^\varepsilon|^2 \rho_\tau^\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau \\ \lesssim& |t-s| \int_s^t \int_\Omega \langle \nabla u_\tau^\varepsilon, B_\varepsilon^{-1} \nabla u_\tau^\varepsilon \rangle \rho_\tau^\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned} \tag{3.28}$$

This gives the equi-continuity of ρ_t^ε in $(\mathcal{P}(\Omega), W_2)$.

Second, for any t fixed, as $\int_\Omega \rho_t^\varepsilon \,\mathrm{d}x = 1$ and Ω is compact, by [[ABS$^+$21],Theorem 8.8], the weak$*$ convergence of $\rho_t^\varepsilon \in \mathcal{P}$ to some $\rho_t \in \mathcal{P}$ implies that

$$W_2(\rho_t^\varepsilon, \rho_t) \to 0. \tag{3.29}$$

We then complete the proof by applying the Arzelá–Ascoli Theorem in $(\mathcal{P}(\Omega), W_2)$. □

4. Passing limit in EDI formulation of ε-gradient flow

In this section, we prove that the EDI formulation (2.21) of ε-gradient flow (2.15) converges to the limiting EDI (2.26). To this end, we need to prove three lower bounds for the functionals (2.1), (2.16), and (2.17) on the left-hand-side of (2.21). Recall the definitions of $\bar{E}, \psi, \psi^*$ in Section 2.4. The lower bounds estimates are stated in the following.

Theorem 4.1. *Assume the initial data ρ_0^ε satisfies the assumptions of Lemma 3.5. Let further ρ_0 be the limit of ρ_0^ε in $(\mathcal{P}(\Omega), W_2)$ and ρ_0^ε be well-prepared in the sense of (2.36). Then*

(i) there exists a subsequence ρ^ε and $\rho \in C([0,T]; L^2(\Omega))$ such that (3.25) holds;

(ii) for a.e. $t \in [0,T]$, the lower bound for free energy holds

$$\liminf_{\varepsilon \to 0} E_\varepsilon(\rho_t^\varepsilon) \geq \overline{E}(\rho_t); \tag{4.1}$$

(iii) for any $t \in [0,T]$, the lower bound for the dissipation on the cotangent plane holds

$$\liminf_{\varepsilon \to 0} \int_0^t \psi_\varepsilon^* \left(\rho_\tau^\varepsilon, -\frac{\delta E_\varepsilon}{\delta \rho}(\rho_\tau^\varepsilon)\right) \,\mathrm{d}\tau \geq \int_0^t \psi^* \left(\rho_\tau, -\frac{\delta \overline{E}}{\delta \rho}(\rho_\tau)\right) \,\mathrm{d}\tau; \tag{4.2}$$

(iv) for any $t \in [0,T]$, the lower bound for the dissipation on the tangent plane holds

$$\liminf_{\varepsilon \to 0} \int_0^t \psi_\varepsilon(\rho_\tau^\varepsilon, \partial_\tau \rho_\tau^\varepsilon) \,\mathrm{d}\tau \geq \int_0^t \psi(\rho_\tau, \partial_\tau \rho_\tau) \,\mathrm{d}\tau. \tag{4.3}$$

As mentioned before, our approach relies on the idea of convergence of functionals in a variational setting. In particular, we make use of the following result which is a special case of by now classical results of Γ-convergence. See for example, [35, Theorems 4.1, 4.4], and also [13, 12, 18] for more detailed explanations.

Theorem 4.2 *(Γ-conv). Let Ω be an open bounded domain of $\mathbb{R}^n$ and $A_\varepsilon(\cdot) = A(\cdot, \frac{\cdot}{\varepsilon})$ be a symmetric positive definite matrix. Consider the functional*

$$\mathcal{F}_\varepsilon(v) = \int_\Omega \left\langle A\left(x, \frac{x}{\varepsilon}\right) \nabla v, \nabla v \right\rangle \,\mathrm{d}x, \quad v \in H_0^1(\Omega) + w \tag{4.4}$$

where $w \in H^1(\Omega)$ is given. Then $\mathcal{F}_\varepsilon$ Γ-converges in $L^2(\Omega)$ to the following functional

$$\mathcal{F}(v) = \int_\Omega \left\langle \overline{A}(x) \nabla v, \nabla v \right\rangle \,\mathrm{d}x, \quad v \in H_0^1(\Omega) + w. \tag{4.5}$$

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In detail,

(1) for any $v_\varepsilon \in H_0^1(\Omega) + w$ *that converges to* $v \in H_0^1(\Omega) + w$ *in* $L^2(\Omega)$*, it holds that*

$$\liminf_{\varepsilon \to 0} \mathcal{F}_\varepsilon(v_\varepsilon) \geq \mathcal{F}(v); \tag{4.6}$$

(2) for any $v \in H_0^1(\Omega) + w$*, there exists* $v_\varepsilon \in H_0^1(\Omega) + w$ *that converges to* v *in* $L^2(\Omega)$*, such that*

$$\lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon(v_\varepsilon) = \mathcal{F}(v). \tag{4.7}$$

Furthermore, the effective matrix $\overline{A}$ *can be found by the following variational formula: for any* $p \in \mathbb{R}^n$*,*

$$\langle \overline{A}(x)p, p \rangle = \inf \left\{ \int_{\mathbb{T}^n} \langle A\,(x, y)\,(\nabla v + p),\ (\nabla v + p) \rangle \ \mathrm{d}y, \quad v \in H^1(\mathbb{T}^n) \right\}. \tag{4.8}$$

As an application, we will apply the above result to the case $\Omega = \mathbb{T}^n$ and

$$A(x, y) = D(x, y)\ \left(= \pi(x, y) B^{-1}(y) \right) \quad \text{(see (A.1))}.$$

The resultant formula for $\overline{A}(x)$ is given by $\overline{D} + \overline{G}$; see the expressions of $\overline{D}$ and $\overline{G}$ in (A.8). In Appendix A, we derive the same formula using asymptotic analysis.

Proof of (4.1). This statement follows directly from [2, Lemma 9.4.3] which says that the entropy functional is jointly lower-semicontinuous with respect to the weak convergence of ρ_t^ε and π_ε. In our case, it also follows simply from the strong convergence of f_t^ε (together with the fact that f_t^ε is uniformly bounded from above and away from zero):

$$\lim_{\varepsilon \to 0} \int_\Omega \rho_t^\varepsilon \log \frac{\rho_t^\varepsilon}{\pi_\varepsilon} \ \mathrm{d}x = \lim_{\varepsilon \to 0} \int_\Omega f_t^\varepsilon (\log f_t^\varepsilon) \pi_\varepsilon \ \mathrm{d}x = \int_\Omega f_t (\log f_t) \overline{\pi} \ \mathrm{d}x = \int_\Omega \rho_t \log \frac{\rho_t}{\overline{\pi}} \ \mathrm{d}x. \qquad \square$$

Proof of (4.2) (time independence case). Let $\tau \in [0, T]$ be fixed. We will prove that

$$\liminf_{\varepsilon \to 0} \psi_\varepsilon^*(\rho_\tau^\varepsilon, -\log \frac{\rho_\tau^\varepsilon}{\pi_\varepsilon}) \geq \psi^* \left(\rho_\tau, -\log \frac{\rho_\tau}{\overline{\pi}} \right). \tag{4.9}$$

We re-write the functional ψ^* in the following way,

$$\begin{aligned} \psi_\varepsilon^*(\rho_\tau^\varepsilon, -\log \frac{\rho_\tau^\varepsilon}{\pi_\varepsilon}) =& \frac{1}{2} \int_\Omega \left\langle \nabla \log \frac{\rho_\tau^\varepsilon}{\pi_\varepsilon}, B_\varepsilon^{-1} \nabla \log \frac{\rho_\tau^\varepsilon}{\pi_\varepsilon} \right\rangle \rho_\tau^\varepsilon \ \mathrm{d}x \\ =& 2 \int_\Omega \left\langle \nabla \sqrt{\frac{\rho_\tau^\varepsilon}{\pi_\varepsilon}}, B_\varepsilon^{-1} \pi_\varepsilon \nabla \sqrt{\frac{\rho_\tau^\varepsilon}{\pi_\varepsilon}} \right\rangle \mathrm{d}x \\ =& 2 \int_\Omega \left\langle \nabla w_\tau^\varepsilon, D_\varepsilon \nabla w_\tau^\varepsilon \right\rangle \mathrm{d}x, \end{aligned}$$

where

$$w_\tau^\varepsilon := \sqrt{f_\tau^\varepsilon}, \quad \text{and} \quad D_\varepsilon = B_\varepsilon^{-1} \pi_\varepsilon.$$

As $f_\tau^\varepsilon \to f_\tau = \frac{\rho_\tau}{\overline{\pi}}$ strongly in $L^p(\Omega)$ for any $p \geq 1$, we have $w_\tau^\varepsilon \to w_\tau := \sqrt{f_\tau} = \sqrt{\frac{\rho_\tau}{\overline{\pi}}}$ in $L^2(\Omega)$. Now we can invoke Theorem 4.2 to deduce that

$$\begin{aligned} &\liminf_{\varepsilon \to 0} 2 \int_\Omega \left\langle \nabla w_\tau^\varepsilon, D_\varepsilon \nabla w_\tau^\varepsilon \right\rangle \mathrm{d}x \\ &\geq 2 \int_\Omega \langle \nabla w_\tau, (\overline{D} + \overline{G}) \nabla w_\tau \rangle \ \mathrm{d}x = 2 \int_\Omega \left\langle \nabla \sqrt{f_\tau}, (\overline{D} + \overline{G}) \nabla \sqrt{f_\tau} \right\rangle \ \mathrm{d}x \\ &= 2 \int_\Omega \left\langle \nabla \sqrt{\frac{\rho_\tau}{\overline{\pi}}}, (\overline{D} + \overline{G}) \nabla \sqrt{\frac{\rho_\tau}{\overline{\pi}}} \right\rangle \ \mathrm{d}x = \frac{1}{2} \int_\Omega \left\langle \nabla \log \frac{\rho_\tau}{\overline{\pi}}, \left(\frac{\overline{D} + \overline{G}}{\overline{\pi}} \right) \nabla \log \frac{\rho_\tau}{\overline{\pi}} \right\rangle \rho_\tau \ \mathrm{d}x \\ &= \frac{1}{2} \int_\Omega \left\langle \nabla \log \frac{\rho_\tau}{\overline{\pi}}, \overline{B}^{-1} \nabla \log \frac{\rho_\tau}{\overline{\pi}} \right\rangle \rho_\tau \ \mathrm{d}x = \psi^* \left(\rho_\tau, -\log \frac{\rho_\tau}{\overline{\pi}} \right), \end{aligned}$$

concluding the result (4.9), with the identification $\overline{B} = \left(\frac{\overline{D}+\overline{G}}{\pi}\right)^{-1}$, from (A.9). □

Proof of (4.3) (time independence case). Here we establish

$$\liminf_{\varepsilon\to 0} \psi_\varepsilon(\rho^\varepsilon, s^\varepsilon) \geq \psi(\rho, s) \tag{4.10}$$

for any $\rho^\varepsilon \rightharpoonup \rho$ in $L^1(\Omega)$ and $s^\varepsilon \rightharpoonup s$ in $L^2(\Omega)$ with the property that

$$f^\varepsilon = \frac{\rho^\varepsilon}{\pi_\varepsilon} \longrightarrow f = \frac{\rho}{\pi} \quad \text{in} \quad L^2(\Omega).$$

Using the definition of ψ_ε, we have

$$\psi_\varepsilon(\rho^\varepsilon, s^\varepsilon) = \sup_{\xi\in L^2(\Omega)} \left\{ \int_\Omega \xi s^\varepsilon \, \mathrm{d}x - \frac{1}{2} \int_\Omega \langle \nabla\xi, B_\varepsilon^{-1}\nabla\xi\rangle \rho^\varepsilon \, \mathrm{d}x \right\} \tag{4.11}$$

and likewise,

$$\psi(\rho, s) = \sup_{\xi\in L^2(\Omega)} \left\{ \int_\Omega \xi s \, \mathrm{d}x - \frac{1}{2} \int_\Omega \langle \nabla\xi, \bar{B}^{-1}\nabla\xi\rangle \rho \, \mathrm{d}x \right\}. \tag{4.12}$$

Note that the supremum in both definitions can be attained. In particular, there is a $\tilde{\xi}$ such that

$$\psi(\rho, s) = \int_\Omega \tilde{\xi} s \, \mathrm{d}x - \frac{1}{2} \int_\Omega \langle \nabla\tilde{\xi}, \bar{B}^{-1}\nabla\tilde{\xi}\rangle \rho \, \mathrm{d}x \quad \text{where} \quad s = -\nabla\cdot\left(\rho \bar{B}^{-1}\nabla\tilde{\xi}\right). \tag{4.13}$$

Next we make use of an approximating sequence $\tilde{\xi}^\varepsilon \rightharpoonup \tilde{\xi}$ in $H^1(\Omega)$ (and hence $\tilde{\xi}^\varepsilon \to \tilde{\xi}$ in $L^2(\Omega)$) such that

$$\lim_{\varepsilon\to 0} \frac{1}{2} \int_\Omega \langle \nabla\tilde{\xi}^\varepsilon, B_\varepsilon^{-1}\nabla\tilde{\xi}^\varepsilon\rangle \rho_\varepsilon \, \mathrm{d}x = \frac{1}{2} \int_\Omega \langle \nabla\tilde{\xi}, \overline{B}^{-1}\nabla\tilde{\xi}\rangle \rho \, \mathrm{d}x. \tag{4.14}$$

The above is equivalent to

$$\lim_{\varepsilon\to 0} \frac{1}{2} \int_\Omega \langle \nabla\tilde{\xi}^\varepsilon, D_\varepsilon\nabla\tilde{\xi}^\varepsilon\rangle f^\varepsilon \, \mathrm{d}x = \frac{1}{2} \int_\Omega \langle \nabla\tilde{\xi}, (\overline{D}+\overline{G})\nabla\tilde{\xi}\rangle f \, \mathrm{d}x. \tag{4.15}$$

The construction of $\tilde{\xi}^\varepsilon$ can essentially be given by Theorem 4.2 if we set $A_\varepsilon = D_\varepsilon f^\varepsilon$. But in order to separate the dependence between D_ε and f^ε, a different argument is needed. We will provide the details in Appendix B.

Now by the fact that $\tilde{\xi}^\varepsilon \longrightarrow \tilde{\xi}$ in $L^2(\Omega)$, together with the assumption $s^\varepsilon \rightharpoonup s$ in $L^2(\Omega)$, we have

$$\int_\Omega \tilde{\xi}^\varepsilon s^\varepsilon \, \mathrm{d}x \longrightarrow \int_\Omega \tilde{\xi} s \, \mathrm{d}x.$$

Then (4.15) implies that

$$\begin{aligned}
\psi(\rho, s) &= \int_\Omega \tilde{\xi} s \, \mathrm{d}x - \frac{1}{2} \int_\Omega \langle \nabla\tilde{\xi}, \overline{B}^{-1}\nabla\tilde{\xi}\rangle \rho \, \mathrm{d}x \\
&= \lim_{\varepsilon\to 0} \left\{ \int_\Omega \tilde{\xi}^\varepsilon s^\varepsilon \, \mathrm{d}x - \frac{1}{2} \int_\Omega \langle \nabla\tilde{\xi}^\varepsilon, B_\varepsilon^{-1}\nabla\tilde{\xi}^\varepsilon\rangle \rho^\varepsilon \, \mathrm{d}x \right\} \\
&\leq \liminf_{\varepsilon\to 0} \left[\sup_\xi \left\{ \int_\Omega \xi s^\varepsilon \, \mathrm{d}x - \frac{1}{2} \int_\Omega \langle \nabla\xi, B_\varepsilon^{-1}\nabla\xi\rangle \rho^\varepsilon \, \mathrm{d}x \right\} \right] \\
&\leq \liminf_{\varepsilon\to 0} \psi(\rho^\varepsilon, s^\varepsilon),
\end{aligned} \tag{4.16}$$

which completes the proof for (4.16). □

Proof of (4.2) and (4.3): time dependent case. To extend the time independent case to the time dependent case and finish the proofs of lower bounds (4.2) and (4.3), we will make use of a general Γ-lim inf result as stated in [51, Cor. 4.4]. Specifically, let H be a separable and reflexive Banach space, and g_n,

$g_\infty : (0,T)\times H \longrightarrow (-\infty,\infty]$ be such that $g_n(t,\cdot)$ and $g_\infty(t,\cdot): H \longrightarrow (-\infty,\infty]$ are convex and for all $u\in H$ and a.e. $t\in(0,T)$, the following holds:

$$g_\infty(t,u) \le \inf\left\{\liminf_n g_n(t,u_n) : u_n \rightharpoonup u \ \text{ in } H\right\}. \tag{4.17}$$

Then for $p\in[1,\infty]$, $u_n \rightharpoonup u$ in $L^p(0,T;H)$ (weak-$*$ if $p=\infty$) and $t\longrightarrow \max\{0,-g_n(t,u_n(t)\}$ uniformly integrable, we have,

$$\int_0^T g_\infty(t,u(t))\,\mathrm{d}t \le \liminf_n \int_0^T g_n(t,u_n(t))\,\mathrm{d}t. \tag{4.18}$$

Note that the uniform integrability condition is automatically satisfied if g_n are non-negative, or bounded from below. See also the remark after Cor. 4.4 in [51].

For (4.2), we set $H=H^1(\Omega)$,

$$g^*_\varepsilon(t,w) := 2\int_\Omega \langle\nabla w, D_\varepsilon\nabla w\rangle\,\mathrm{d}x, \quad\text{and}\quad g^*_\infty(t,w) := 2\int_\Omega\langle\nabla w,(\overline{D}+\overline{G})\nabla w\rangle\,\mathrm{d}x.$$

Then $g^*_\varepsilon(t,\cdot)$ and $g^*_\infty(t,\cdot)$ are convex and (4.17) holds true by the time independent version of (4.2). Hence we have

$$\int_0^T 2\int_\Omega\langle\nabla w,(\overline{D}+\overline{G})\nabla w\rangle\,\mathrm{d}x\,\mathrm{d}t \le \liminf_\varepsilon \int_0^T 2\int_\Omega\langle\nabla w^\varepsilon(t), D_\varepsilon\nabla w^\varepsilon(t)\rangle\,\mathrm{d}x\,\mathrm{d}t$$

provided $w^\varepsilon \rightharpoonup w$ in $L^2((0,T);H)$. This last condition is satisfied by the identification $w^\varepsilon(t)=\sqrt{\frac{\rho^\varepsilon(t)}{\pi_\varepsilon}}$, $w(t)=\sqrt{\frac{\rho(t)}{\pi}}$, and (3.20). This concludes the lower bound (4.2).

For (4.3), we set $H=L^2(\Omega)$,

$$g_\varepsilon(t,s) := \psi_\varepsilon(\rho(t),s) = \frac{1}{2}\int_\Omega\langle\nabla u^\varepsilon, B_\varepsilon^{-1}\nabla u^\varepsilon\rangle\rho^\varepsilon(t)\,\mathrm{d}x \quad\text{with}\quad -\nabla\cdot(\rho^\varepsilon B_\varepsilon^{-1}\nabla u^\varepsilon)=s,$$

and

$$g_\infty(t,s) := \psi(\rho(t),s) = \frac{1}{2}\int_\Omega\langle\nabla u, \overline{B}^{-1}\nabla u\rangle\rho(t)\,\mathrm{d}x \quad\text{with}\quad -\nabla\cdot(\rho\overline{B}^{-1}\nabla u)=s.$$

Again, $g_\varepsilon(t,\cdot)$ and $g_\infty(t,\cdot)$ are convex because the map $s\to u^\varepsilon$ or u is uniquely defined and linear. By (4.10), (4.17) is satisfied. Hence, we have

$$\int_0^T \psi\big(\rho(t),s(t)\big)\,\mathrm{d}t \le \liminf \int_0^T \psi_\varepsilon\big(\rho^\varepsilon(t),s^\varepsilon(t)\big)\,\mathrm{d}t$$

upon the identification $s^\varepsilon(t)=\partial_t\rho^\varepsilon_t$ and $s(t)=\partial_t\rho_t$. The fact that $s^\varepsilon\rightharpoonup s$ in $L^2((0,T);H)$ follows from (3.23). Lower bound (4.3) is thus proved.

The above conclude the proof for Theorem 4.1. □

5. Comparison between limiting Wasserstein distances

In this section, we use the just established convergence result for gradient flows in EDI form to further analyse the induced limiting Wasserstein distance $\overline{W}$. In particular, we will show that the limiting Wasserstein metric $\overline{W}$ is in general, different, and in fact *strictly larger than* W_{GH} obtained from the Gromov–Hausdorff limit of W_ε which is a commonly considered mode of convergence of metric spaces. Gromov–Hausdorff distance can be used to compare the distortion of two metric spaces from being isometric. The particular property needed in this paper is that the Gromov–Hausdorff convergence of a compact metric space Ω_k implies the Gromov–Hausdorff convergence of the Wasserstein space $(\mathcal{P}(\Omega_k), W_k)$ [53, Theorem 28.6]. Briefly stated, let $(\mathcal{X}, d_\mathcal{X})$ and $(\mathcal{Y}, d_\mathcal{Y})$ be two metric spaces. Their

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Gromov–Hausdorff distance is defined as [53, (27.2)]

$$D_{GH}(\mathcal{X},\mathcal{Y}) = \frac{1}{2}\inf_{\mathcal{R}} \sup_{(x,y),(x',y')\in\mathcal{R}} \left| d_{\mathcal{X}}(x,x') - d_{\mathcal{Y}}(y,y') \right|, \tag{5.1}$$

where $\mathcal{R} \subset \mathcal{X}\times\mathcal{Y}$ is a correspondence or relation between $\mathcal{X}$ and $\mathcal{Y}$. We refer to [53, Chapters 27, 28] for more detailed information about the concept of Gromov–Hausdorff distances and convergence. For our application, we will take $(\mathcal{X}, d_{\mathcal{X}}) := (\Omega, d_\varepsilon)$ or $(\mathcal{P}(\Omega), W_\varepsilon)$.

We remark that several of the following statements require the existence of densities (with respect to Lebesgue measure) for the underlying probability measures and the space to be geodesic complete. These are automatically satisfied by our standing assumptions (see Section 2.4).

5.1. Effective Wasserstein distance $\overline{W}$ induced by convergence of gradient flows

For convenience, we recall here the Kantorovich and Benamou–Brenier formulations (1.4) and (1.5) for our ε-Wasserstein metric W_ε:

$$W_\varepsilon^2(\rho_0,\rho_1) := \inf\left\{\iint d_\varepsilon^2(x,y)\,\mathrm{d}\gamma(x,y);\quad \int_\Omega \gamma(x,\,\mathrm{d}y) = \rho_0(x)\,\mathrm{d}x,\ \int_\Omega \gamma(\,\mathrm{d}x,y) = \rho_1(y)\,\mathrm{d}y\right\} \tag{5.2}$$

and

$$W_\varepsilon^2(\rho_0,\rho_1) := \inf\left\{\int_0^1\int \rho_t(x)\langle B_\varepsilon(x)v_t(x), v_t(x)\rangle\,\mathrm{d}x\,\mathrm{d}t,\quad (\rho_t,v_t)\in V(\rho_0,\rho_1)\right\}, \tag{5.3}$$

where V is defined in (1.6). The ε-metric d_ε on $\Omega\subset\mathbb{R}^n$ is given via the least action

$$d_\varepsilon^2(x,y) := \inf\left\{\int_0^1 \langle B_\varepsilon(z_t)\dot{z}_t,\dot{z}_t\rangle\,\mathrm{d}t,\quad z_0 = x,\quad z_1 = y\right\}. \tag{5.4}$$

A curve $z(\,\cdot\,)\in AC([0,1];\mathbb{R}^n)$ that achieves the infimum in (5.4) is a geodesic in the metric space $(\mathbb{R}^n, d_\varepsilon)$. From [6, Theorem A,B], (5.2) and (5.3) are equivalent.

The same formulations hold for our induced limit Wasserstein distance $\overline{W}$. More precisely, we have

$$\overline{W}^2(\rho_0,\rho_1) := \inf\left\{\int\int \overline{d}^2(x,y)\,\mathrm{d}\gamma(x,y);\quad \int_\Omega \gamma(x,\,\mathrm{d}y) = \rho_0(x)\,\mathrm{d}x,\ \int_\Omega \gamma(\,\mathrm{d}x,y) = \rho_1(y)\,\mathrm{d}y\right\}, \tag{5.5}$$

and the equivalent formulation

$$\overline{W}^2(\rho_0,\rho_1) := \inf\left\{\int_0^1\int \rho_t(x)\langle \overline{B}v_t(x), v_t(x)\rangle\,\mathrm{d}x\,\mathrm{d}t,\quad (\rho_t,v_t)\in V(\rho_0,\rho_1)\right\}. \tag{5.6}$$

Here the constant matrix $\overline{B}$ is defined in (A.9) and the induced-metric $\overline{d}$ on $\Omega\subset\mathbb{R}^n$ is again given via the least action

$$\overline{d}^2(x,y) := \inf\left\{\int_0^1 \langle \overline{B}\dot{z}_t,\dot{z}_t\rangle\,\mathrm{d}t,\quad z_0 = x,\quad z_1 = y\right\}. \tag{5.7}$$

From the Euler–Lagrangian equation for the minimiser of (5.7), the optimal curve $\tilde{z}(\,\cdot\,)$ that achieves the least action satisfies $\overline{B}\ddot{\tilde{z}}_t = 0$, and hence it has constant speed, $\dot{\tilde{z}}_t = y - x$. Thus, we have explicitly

$$\overline{d}^2(x,y) = \langle \overline{B}(y-x), y-x\rangle = \langle \overline{B}\hat{n},\hat{n}\rangle |y-x|^2,\quad \text{where}\quad \hat{n} = \frac{y-x}{|y-x|}. \tag{5.8}$$

Note that both W^ε and $\overline{W}$ induce a Riemannian metric on $\mathcal{P}(\Omega)$. More precisely, for any $\rho\in\mathcal{P}(\Omega)$, and any $s_1, s_2 \in T_{\mathcal{P}}$, the tangent plane at ρ, the first fundamental form are defined, respectively, as

$$\langle s_1, s_2\rangle_{T_{\mathcal{P}},T_{\mathcal{P}},\varepsilon} := \int \rho(x)\langle B_\varepsilon^{-1}(x)\nabla u_1(x), \nabla u_2(x)\rangle\,\mathrm{d}x, \tag{5.9}$$

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where $s_i = -\nabla \cdot (\rho B_\varepsilon^{-1} \nabla u_i)$, $i = 1, 2$ for W^ε, and

$$\langle s_1, s_2 \rangle_{T_\mathcal{P}, T_\mathcal{P}} := \int \rho(x) \langle \overline{B}^{-1}(x) \nabla u_1(x), \nabla u_2(x) \rangle \, \mathrm{d}x, \tag{5.10}$$

where $s_i = -\nabla \cdot (\rho \overline{B}^{-1} \nabla u_i)$, $i = 1, 2$ for $\overline{W}$. This is also manifested by the fact that both the corresponding dissipation functionals are *bilinear forms* in s:

$$\psi_\varepsilon(\rho, s) = \frac{1}{2} \int_\Omega \langle \nabla u, B_\varepsilon^{-1} \nabla u \rangle \rho \, \mathrm{d}x \quad \text{with } s = -\nabla \cdot \left(\rho B_\varepsilon^{-1} \nabla u \right),$$

and

$$\psi(\rho, s) = \frac{1}{2} \int_\Omega \langle \nabla u, \overline{B}^{-1} \nabla u \rangle \rho \, \mathrm{d}x \quad \text{with } s = -\nabla \cdot \left(\rho \overline{B}^{-1} \nabla u \right).$$

5.2. *The Gromov–Hausdorff limit* W_{GH} *of* W_ε

Now we consider the convergence in the Gromov–Hausdorff sense of W_ε to a limiting Wasserstein metric, denoted as W_{GH}.

We first show that even in one dimension, in general it is always the case that $W_{\mathrm{GH}} < \overline{W}$ unless π_ε and B_ε are related to each other in some specific way. Recall the metric d_ε in (5.4). From the Euler–Lagrangian equation for the minimiser $z_t = \tilde{z}_t^\varepsilon$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(2B_\varepsilon(z_t)\dot{z}_t) = B_\varepsilon'(z_t)(\dot{z}_t)^2,$$

leading to $B_\varepsilon'(z_t)\dot{z}_t^2 + 2B_\varepsilon(z_t)\ddot{z}_t = 0$ and thus

$$B_\varepsilon(z)\dot{z}^2 = C_\varepsilon(x, y), \qquad \text{for some constant } C_\varepsilon(x, y).$$

Upon solving this ODE for z_t with the two boundary conditions $z(0) = x$, $z(1) = y$, we have

$$\sqrt{C_\varepsilon(x, y)} = \int_x^y \sqrt{B_\varepsilon(z)} \, \mathrm{d}z.$$

Hence the infimum in (5.4) is given by

$$d_\varepsilon^2(x, y) = C_\varepsilon(x, y) = \left(\int_x^y \sqrt{B_\varepsilon(z)} \, \mathrm{d}z \right)^2. \tag{5.11}$$

As $B_\varepsilon(x) = B(\frac{x}{\varepsilon})$, it is easy to verify that for any $x, y \in \Omega$, there exist some integer N_ε and $\delta \in (-1, 1)$, such that $y - x = N_\varepsilon \varepsilon + \delta\varepsilon$ and $N_\varepsilon \varepsilon \to |x - y|$. Notice also $B(\cdot)$ is 1-periodic. Hence,

$$d_\varepsilon^2(x, y) = \left(\varepsilon \int_{\frac{x}{\varepsilon}}^{\frac{y}{\varepsilon}} \sqrt{B(s)} \, \mathrm{d}s \right)^2 = \left(\varepsilon N_\varepsilon \int_0^1 \sqrt{B(s)} \, \mathrm{d}s + \varepsilon \int_0^\delta \sqrt{B(s)} \, \mathrm{d}s \right)^2$$
$$\overrightarrow{\varepsilon \to 0} \; |x - y|^2 \left(\int_0^1 \sqrt{B(s)} \, \mathrm{d}s \right)^2 =: d_{\mathrm{GH}}^2(x, y).$$

Notice that if one chooses $\mathcal{R}$ to be the identity map as the correspondence between the metric spaces $\mathcal{X} := (\Omega, d_\varepsilon)$ and $\mathcal{Y} := (\Omega, d_{\mathrm{GH}})$, then from (5.1), we have

$$D_{\mathrm{GH}}(\mathcal{X}, \mathcal{Y}) \le \frac{1}{2} \sup_{(x,x),(y,y) \in \mathcal{X} \times \mathcal{Y}} |d_\varepsilon(x, y) - d_{\mathrm{GH}}(x, y)| \to 0.$$

Hence the one dimensional metric space (Ω, d_ε) Gromov–Hausdorff converges to $(\Omega, d_{\mathrm{GH}})$. By [53, Theorem 28.6], the Wasserstein distance W_ε defined in (5.2) also converges to the following limiting

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Wasserstein distance W_{GH} in the Gromov–Hausdorff sense,

$$W_{\mathrm{GH}}^2(\rho_0, \rho_1) := \inf\left\{\int\int d_{\mathrm{GH}}^2(x,y)\,\mathrm{d}\gamma(x,y);\quad \int_\Omega \gamma(x,\,\mathrm{d}y) = \rho_0(x)\,\mathrm{d}x,\ \int_\Omega \gamma(\,\mathrm{d}x, y) = \rho_1(y)\,\mathrm{d}y\right\}. \tag{5.12}$$

Again by [6, Theorem AB], W_{GH} can be equivalently written in the Benamou–Brenier formulation

$$W_{\mathrm{GH}}^2(\rho_0, \rho_1) := \inf\left\{\int_0^1\int \rho_t(x)\langle \overline{C}v_t(x), v_t(x)\rangle\,\mathrm{d}x\,\mathrm{d}t,\quad (\rho_t, v_t)\in V(\rho_0,\rho_1)\right\} \tag{5.13}$$

with $\overline{C} = \left(\int_0^1 \sqrt{B(s)}\,\mathrm{d}s\right)^2$.

On the other hand, in one dimension, we can solve the cell problem (A.6) explicitly:

$$\partial_y\big(D(x,y)\partial_y w(x,y)\big) = -\partial_y\left(D(x,y)\right), \quad \text{where}\ \ D(x,y) = \pi(x,y)B(y)^{-1},$$
$$\partial_y w(x,y) = -1 + \frac{C(x)}{D(x,y)} \quad \text{with}\ \ C(x) = \left(\int \frac{1}{D(x,y)}\,\mathrm{d}y\right)^{-1}.$$

Then (A.7) and (A.9) are given as

$$\overline{D}(x) = \int D(x,y)\,\mathrm{d}y,$$
$$\overline{G}(x) = \int D(x,y)\left(-1+\frac{C(x)}{D(x,y)}\right)\,\mathrm{d}y = -\int D(x,y)\,\mathrm{d}y + \left(\int\frac{1}{D(x,y)}\,\mathrm{d}y\right)^{-1},$$
$$\overline{B} = \left(\frac{\overline{D}+\overline{G}}{\overline{\pi}}\right)^{-1} = \overline{\pi}\int\frac{1}{D(x,y)}\,\mathrm{d}y = \overline{\pi}\int\frac{B(y)}{\pi(x,y)}\,\mathrm{d}y.$$

By the Cauchy–Schwarz inequality, we always have

$$\overline{C} = \left(\int_0^1\sqrt{B(s)}\,\mathrm{d}s\right)^2 = \left(\int_0^1\sqrt{\pi(x,y)}\sqrt{\frac{B(y)}{\pi(x,y)}}\,\mathrm{d}y\right)^2$$
$$\le \left(\int \pi(x,y)\,\mathrm{d}y\right)\left(\int\frac{B(y)}{\pi(x,y)}\,\mathrm{d}y\right) = \overline{B}(x),$$

and the equality holds if and only if there exists some constant $c>0$ such that

$$\sqrt{\pi(x,y)} = c\sqrt{\frac{B(y)}{\pi(x,y)}},\quad \text{i.e.}\quad \pi(x,y) = \pi(y) = c\sqrt{B(y)}. \tag{5.14}$$

Hence, unless $\pi(y) = c\sqrt{B(y)}$, we always have

$$d_{\mathrm{GH}}(x,y) < \overline{d}(x,y)\quad \text{for all}\ x,y\in\Omega$$

i.e. $W_{\mathrm{GH}} < \overline{W}$. As an afterthought, it seems not quite surprising that some condition, such as (5.14), is needed in order for $\overline{W}$ to be equal to W_{GH}. We will elaborate upon this at the end of this section.

Next, we illustrate the n-dimensional case by means of an example. From [12, Section 3.3], it is shown that the functional

$$\mathcal{F}_\varepsilon(z) = \int_0^1 \langle B_\varepsilon(z_t)\dot{z}_t, \dot{z}_t\rangle\,\mathrm{d}t,\quad \text{for}\quad z(\,\cdot\,)\in(H^1([0,1]))^n\quad\text{with}\quad z_0 = x,\quad z_1 = y, \tag{5.15}$$

Γ-converges with respect to the strong $L^2(0,1)$-topology to

$$\mathcal{F}(z) = \int_0^1 \varphi(\dot{z}(t))\,\mathrm{d}t\quad\text{for}\quad z(\,\cdot\,)\in(H^1([0,1]))^n,\quad\text{with}\quad z_0 = x,\quad z_1 = y, \tag{5.16}$$

where the limiting integrand φ is given by

$$\varphi(v) := \lim_{T\to+\infty} \inf_{u\in(H_0^1([0,T]))^n} \left\{ \frac{1}{T}\int_0^T \langle B(u(t)+vt)(\dot u(t)+v),\ \dot u(t)+v\rangle\, \mathrm{d}t \right\}. \tag{5.17}$$

Now following [12, Example 3.3], we consider $B_\varepsilon(z) = b(\frac{z}{\varepsilon})$ where b is the following 1-periodic function on $[0, 1]^n$,

$$b(y) = \begin{cases} \beta & \text{if } y \in (0,1)^n; \\ \alpha & \text{if for some } i,\ y_i \in \mathbb{Z}. \end{cases}$$

If $n\alpha < \beta$, one obtains that the limiting energy integrand φ is given by

$$\varphi(v) = \alpha \left(\sum_{i=1}^n |v_i| \right)^2. \tag{5.18}$$

Using the property of Γ-convergence [12],Theorem 1.21], we deduce also the convergence of the minimum value d_ε^2 of $\mathcal{F}_\varepsilon$ to the minimum value d_{GH}^2 of $\mathcal{F}$, where

$$d_{\mathrm{GH}}(x,y) = \sqrt{\alpha}\left(\sum_{i=1}^n |\hat n_i| \right) |y-x| = \sqrt{\alpha}\|y-x\|_{\ell^1} \quad \text{with } \hat n = \tfrac{y-x}{|y-x|}. \tag{5.19}$$

On the other hand, note that the value α is attained only on the $(n-1)$-dimensional set $\bigcup_{i=1}^n \{y_i \in \mathbb{Z}\}$. This set is *invisible* by $\overline{B}$ which is obtained by solving the elliptic cell problem (A.6). Hence the induced limiting Wasserstein distance $\overline{W}$ (5.5) with $\overline{d}$ defined in (5.7) is $\overline{d}(x,y) = \beta|x-y|$ for all $x, y \in \Omega$. Thus, for this example, we have

$$d_{\mathrm{GH}}(x,y) = \sqrt{\alpha}\|y-x\|_{\ell^1} \le \sqrt{\alpha n}\|y-x\|_{\ell^2} < \sqrt{\beta}|y-x| = \overline{d}(x,y).$$

Hence we have again $W_{\mathrm{GH}} < \overline{W}$.

We would like to point out that for the above example, the integrand φ in (5.17) is always quadratic, or homogeneous of degree 2 in p. (In fact, for any $\lambda \neq 0$, by applying the change of variables $\tilde t = \lambda t$, $\tilde u(\tilde t) = u(t)$, it is easy to verify that $\varphi(\lambda v) = \lambda^2 \varphi(v)$.) However, the φ in (5.18) is *not bilinear* in p, in contrast to the φ in (5.7):

$$\varphi(p) = \langle \bar B p, p \rangle.$$

Below we give further remarks about the discrepancy between $\overline{W}$ and W_{GH}.

(1) We first explain the condition (5.14). This is nothing but the fact that one can choose the Riemannian metric $(\mathbb{R}, g_\varepsilon)$ with $(g_\varepsilon)_{ij}(x) = B_\varepsilon(x)$, so that the Wasserstein distance on $(\mathbb{R}, g_\varepsilon)$ coincides with W_ε. More precisely, the condition (5.14) implies the volume form on $(\mathbb{R}, g_\varepsilon)$ is

$$\mathrm{dVol} = \sqrt{|g_\varepsilon|}\,\mathrm{d}x = \sqrt{B_\varepsilon}\,\mathrm{d}x = c\pi_\varepsilon(x)\,\mathrm{d}x = c\pi(\frac{x}{\varepsilon})\,\mathrm{d}x. \tag{5.20}$$

Therefore, the heat flow on $(\mathbb{R}, g_\varepsilon)$, in terms of the density function with respect to the volume element dVol is given by

$$\partial_t p_\varepsilon = \frac{1}{\sqrt{|g_\varepsilon|}} \nabla\cdot(\sqrt{|g_\varepsilon|} g_\varepsilon^{ij} \nabla p_\varepsilon) = \frac{1}{\pi_\varepsilon}\nabla\cdot(\pi_\varepsilon B_\varepsilon^{-1}\nabla p_\varepsilon). \tag{5.21}$$

This equation, in terms of the density function $\rho_\varepsilon(x,t) = p_\varepsilon(x,t)\sqrt{|g_\varepsilon|} = p_\varepsilon(x,t)\pi_\varepsilon(x)$, is exactly the W_ε-gradient flow with respect to the relative entropy E_ε in (2.1):

$$\partial_t \rho_\varepsilon = \nabla\cdot(\pi_\varepsilon B_\varepsilon^{-1}\nabla \frac{\rho_\varepsilon}{\pi_\varepsilon}) = \nabla\cdot\left(\rho_t^\varepsilon B_\varepsilon^{-1}\nabla\frac{\delta E_\varepsilon}{\delta\rho}(\rho_t^\varepsilon)\right). \tag{5.22}$$

Therefore, condition (5.14) means that the discrepancy between $\overline{W}$ and W_{GH} does not happen in one dimension when one considers homogenisation of heat flow on $(\mathbb{R}, g_\varepsilon)$. In other words, the homogenised heat flow in one dimension naturally induces the same limiting distance as finding

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the limiting minimum path on $(\mathbb{R}, g_\varepsilon)$. On the other hand, even in one dimension, the convergence of the discrete transport distance to continuous transport distance W_2 requires an isotropic mesh condition [22],eq. (1.3)]. Without this condition, the discrete-to-continuous limiting distance in the Gromov–Hausdorff sense can be different from the continuous transport distance W_2 [22], Theorem 1.1, Remarks 1.2 and 1.3].

(2) We believe that the above conclusion of $W_{\mathrm{GH}} < \overline{W}$ is true in general, particularly in higher dimensions, even if we consider heat flow. This is because the Gromov–Hausdorff limit d_{GH} of d_ε involves finding the minimum or geodesic distance between two points as indicated in (5.4). This amounts to searching for the *minimum path* in the underlying spatial inhomogeneity. On the other hand, the $\overline{B}$ in the limiting induced distance $\overline{d}$ is found by solving an elliptic cell-problem (A.6) which requires taking some *average* of the spatial inhomogeneity. (Note that in contrast, in one dimension, any path will explore the whole inhomogeneous landscape.) Hence, in general d_{GH} and W_{GH} should be smaller than $\overline{d}$ and $\overline{W}$. See also the discussion in [20, p. 4298] and the work [22].

6. Conclusion

This paper provides a variational framework using the energy dissipation inequality to prove the convergence of gradient flows in Wasserstein spaces. Our key contribution is the incorporation of fast oscillations in the underlying energy and medium. In particular, the gradient-flow structure is preserved in the limit but is described with respect to an effective energy and metric. Our result is consistent with asymptotic analysis from the realm of homogenisation. Even though we apply the result to a linear Fokker-Planck equation in a continuous setting, we believe the approach is applicable to a broader class of problems including nonlinear equations or evolutions on graphs and networks.

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Appendix A. Asymptotic analysis for the ε-gradient flow

In this section, we use the method of asymptotic expansion to analyse the convergence of the ε-Fokker-Planck equation (2.4) (or (2.15)) to the limiting homogenised one (2.24).

Recall the assumptions (2.30) and (2.32) for B_ε and π_ε in Section 2.4 and the definition of fast variable $y := \frac{x}{\varepsilon}$. Introducing

$$D(x, y) = \pi(x, y)B^{-1}(y), \tag{A.1}$$

then (3.2) reads

$$\partial_t f^\varepsilon = \frac{1}{\pi_\varepsilon}\nabla\cdot\left(D(x, \frac{x}{\varepsilon})\nabla f^\varepsilon\right). \tag{A.2}$$

Consider the ansatz

$$f^\varepsilon(x,t) = f_0\big(x, \frac{x}{\varepsilon}, t\big) + \varepsilon f_1\big(x, \frac{x}{\varepsilon}, t\big) + O(\varepsilon^2) \quad \text{with } f_0 \text{ and } f_1 \text{ 1-periodic in } y. \tag{A.3}$$

Substituting it into (A.2), we have

$$\partial_t\big(f_0 + \varepsilon f_1 + O(\varepsilon^2)\big) = \frac{1}{\pi(x,y)}\left(\nabla_x + \frac{1}{\varepsilon}\nabla_y\right)\cdot\left(D(x,y)\left(\nabla_x + \frac{1}{\varepsilon}\nabla y\right)\big(f_0 + \varepsilon f_1 + O(\varepsilon^2)\big)\right). \tag{A.4}$$

Terms of different orders are analysed as follows.

(I) $\frac{1}{\varepsilon^2}$-terms. They satisfy,

$$\nabla_y\cdot\big(D(x,y)\nabla_y f_0(x,y,t)\big) = 0.$$

Multiply the above by $f_0(x,y,t)$ and then integrate over y gives $\int |\nabla_y f_0(x,y,t)|^2\,\mathrm{d}y = 0$ which implies $f_0(x,y,t) = f_0(x,t)$.

(II) $\frac{1}{\varepsilon}$-terms. They satisfy,

$$\nabla_y\cdot\big(D(x,y)(\nabla_x f_0 + \nabla_y f_1)\big) = 0. \tag{A.5}$$

For $i = 1, 2, \ldots d$, let $w_i(y)$ be the solution to the cell problem

$$\nabla_y\cdot\big(D(x,y)\nabla_y w_i(x,y)\big) + \nabla_y\cdot(D(x,y)\vec{e}_i) = 0, \tag{A.6}$$

where $\vec{e}_i$ is the unit vector in i-coordinate. The above equation is solvable for each i due to the compatibility condition $\int \nabla_y\cdot(D(x,y)\vec{e}_i)\ \mathrm{d}y = 0$. Then we can write f_1 as

$$f_1(x,y,t) = \sum_i \partial_{x_i} f_0(x,t) w_i(x,y).$$

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(III) $O(1)$-terms. Collecting the $O(1)$-terms in (A.4) and integrating with respect to y lead to

$$\partial_t f_0(x,t)\,\overline{\pi}(x) = \nabla_x \cdot (\overline{D}(x)\nabla_x f_0(x,t)) + \nabla \cdot \left(\sum_i \partial_{x_i} f_0(x,t)\overline{G}_i(x)\right),$$

where

$$\overline{D}(x) := \int \pi(x,y)B^{-1}(y)\,\mathrm{d}y, \quad \overline{G}_i(x) := \int \pi(x,y)B^{-1}(y)\nabla_y w_i(x,y)\,\mathrm{d}y, \tag{A.7}$$

and $\overline{\pi} = \int \pi(x,y)\,\mathrm{d}y$; see (2.33).

Then the leading dynamics in terms of f_0 is given by

$$\partial_t f_0 = \frac{1}{\overline{\pi}}\nabla \cdot \left((\overline{D}+\overline{G})\nabla f_0\right), \quad \text{where } \ \overline{G} = (G_1, G_2, \ldots G_n). \tag{A.8}$$

Upon defining

$$\overline{B}(x) = \left(\frac{\overline{D}+\overline{G}}{\overline{\pi}}\right)^{-1}, \tag{A.9}$$

in terms of $\rho = f_0\overline{\pi}$, (A.8) can be written as

$$\partial_t \rho = \nabla \cdot \left(\rho\,\overline{B}^{-1}\nabla \log \frac{\rho}{\overline{\pi}}\right). \tag{A.10}$$

The above procedure certainly works for the simpler uniform convergence case $\pi_\varepsilon = \pi_\varepsilon^{\mathrm{II}}$ in (2.34) which converges uniformly to π_0. We find it illustrative to write down the homogenized limit equation. In this case, the definition of D (A.1), the cell problem (A.6) and the effective coefficients (A.7) now become

$$D(x,y) = \pi_0(x)B^{-1}(y), \quad \nabla_y \cdot \left(B^{-1}(y)\nabla_y w_i(y)\right) + \nabla_y \cdot \left(B^{-1}(y)\vec{e}_i\right) = 0,$$

and

$$\overline{D}(x) := \pi_0(x)\int B^{-1}(y)\,\mathrm{d}y, \quad \overline{G}(x) := \pi_0(x)\int B^{-1}(y)\nabla_y w(y)\,\mathrm{d}y, \quad (\text{where } \ w = (w_1, w_2 \ldots w_n)),$$

so that

$$\overline{B}(x) = \left(\frac{\overline{D}(x)+\overline{G}(x)}{\pi_0(x)}\right)^{-1} = \left(\int B^{-1}(y)\,\mathrm{d}y + \int B^{-1}(y)\nabla_y w(y)\,\mathrm{d}y\right)^{-1}. \tag{A.11}$$

Then the effective Fokker-Planck equation is given by

$$\partial_t \rho = \nabla \cdot \left(\rho\,\overline{B}^{-1}\nabla \log \frac{\rho}{\pi_0}\right). \tag{A.12}$$

Comparing (A.9) and (A.11), it is clear that there is interaction between B_ε and π_ε in the former case but not in the latter.

Appendix B. Construction of $\tilde{\xi}^\varepsilon$ for (4.15)

Here we construct an approximating sequence $\tilde{\xi}^\varepsilon \rightharpoonup \tilde{\xi}$ in $H^1(\Omega)$ such that (4.15) holds. As mentioned, due to the spatially varying weight function f^ε, in order to decouple the dependence between D_ε and f^ε, an extra step is needed if we want to invoke the classical Γ-convergence result Theorem 4.2. Without loss of generality, we assume that $\tilde{\xi}$ is smooth so that pointwise evaluation $\tilde{\xi}(x)$ is well-defined. This can

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be achieved by first convolving $\tilde{\xi}$ with a smooth kernel. We also recall by statement (1) of Lemma 3.1 that f is a bounded and uniformly positive function.

For this purpose, we write for any $\tilde{\xi}^\varepsilon$ that

$$\begin{aligned}&\frac{1}{2}\int_\Omega \langle\nabla\tilde{\xi}^\varepsilon, D_\varepsilon\nabla\tilde{\xi}^\varepsilon\rangle f^\varepsilon\,\mathrm{d}x\\ &=\frac{1}{2}\int_\Omega \langle\nabla\tilde{\xi}^\varepsilon, D_\varepsilon\nabla\tilde{\xi}^\varepsilon\rangle f_c\,\mathrm{d}x+\frac{1}{2}\int_\Omega \langle\nabla\tilde{\xi}^\varepsilon, D_\varepsilon\nabla\tilde{\xi}^\varepsilon\rangle (f-f_c)\,\mathrm{d}x+\frac{1}{2}\int_\Omega \langle\nabla\tilde{\xi}^\varepsilon, D_\varepsilon\nabla\tilde{\xi}^\varepsilon\rangle (f^\varepsilon-f)\,\mathrm{d}x,\end{aligned}$$

where f_c is some continuous function approximating f. Next, we partition Ω into finitely many cubes C_j and define the following piece-wise constant function

$$\bar{f}_c(x)=\bar{f}_{cj}:=\frac{1}{|C_j|}\int_{C_j} f_c\,\mathrm{d}x \quad \text{for } x\in C_j.$$

Hence

$$\frac{1}{2}\int_\Omega \langle\nabla\tilde{\xi}^\varepsilon, D_\varepsilon\nabla\tilde{\xi}^\varepsilon\rangle f_c\,\mathrm{d}x=\sum_j\frac{1}{2}\int_\Omega \langle\nabla\tilde{\xi}^\varepsilon, D_\varepsilon\nabla\tilde{\xi}^\varepsilon\rangle \bar{f}_{cj}\,\mathrm{d}x+\sum_j\frac{1}{2}\int_\Omega \langle\nabla\tilde{\xi}^\varepsilon, D_\varepsilon\nabla\tilde{\xi}^\varepsilon\rangle (f_c-\bar{f}_{cj})\,\mathrm{d}x.$$

With the above, we have

$$\begin{aligned}&\lim_{\varepsilon\to0}\frac{1}{2}\int_\Omega \langle\nabla\tilde{\xi}^\varepsilon, D_\varepsilon\nabla\tilde{\xi}^\varepsilon\rangle f^\varepsilon\,\mathrm{d}x\\ &=\lim_{\varepsilon\to0}\frac{1}{2}\int_\Omega \langle\nabla\tilde{\xi}^\varepsilon, D_\varepsilon\nabla\tilde{\xi}^\varepsilon\rangle \bar{f}_c\,\mathrm{d}x+\lim_{\varepsilon\to0}\frac{1}{2}\int_\Omega \langle\nabla\tilde{\xi}^\varepsilon, D_\varepsilon\nabla\tilde{\xi}^\varepsilon\rangle (f_c-\bar{f}_c)\,\mathrm{d}x\\ &\quad+\lim_{\varepsilon\to0}\frac{1}{2}\int_\Omega \langle\nabla\tilde{\xi}^\varepsilon, D_\varepsilon\nabla\tilde{\xi}^\varepsilon\rangle (f-f_c)\,\mathrm{d}x+\lim_{\varepsilon\to0}\frac{1}{2}\int_\Omega \langle\nabla\tilde{\xi}^\varepsilon, D_\varepsilon\nabla\tilde{\xi}^\varepsilon\rangle (f^\varepsilon-f)\,\mathrm{d}x.\end{aligned}$$

Now on each C_j, we can invoke Theorem 4.2 to state the existence of recovery sequence $\tilde{\xi}_j^\varepsilon\rightharpoonup\tilde{\xi}$ in $H_0^1(C_j)+g_{cj}$, where $g_{cj}=\tilde{\xi}\big|_{\partial C_j}$ such that

$$\lim_{\varepsilon\to0}\frac{1}{2}\int_{C_j} \langle\nabla\tilde{\xi}_j^\varepsilon, D_\varepsilon\nabla\tilde{\xi}_j^\varepsilon\rangle \bar{f}_{cj}\,\mathrm{d}x=\frac{1}{2}\int_{C_j}\langle\nabla\tilde{\xi},(\overline{D}+\overline{G})\nabla\tilde{\xi}\rangle\bar{f}_{cj}\,\mathrm{d}x. \tag{B.1}$$

Next let $\tilde{\xi}^\varepsilon=\tilde{\xi}_j^\varepsilon$ on C_j. Note that $\tilde{\xi}^\varepsilon$ thus defined is a global H^1-function on Ω. As there are only finitely many cubes C_j, we can conclude that

$$\lim_{\varepsilon\to0}\frac{1}{2}\int_{\Omega} \langle\nabla\tilde{\xi}^\varepsilon, D_\varepsilon\nabla\tilde{\xi}^\varepsilon\rangle \bar{f}_{c}\,\mathrm{d}x=\frac{1}{2}\int_{\Omega}\langle\nabla\tilde{\xi},(\overline{D}+\overline{G})\nabla\tilde{\xi}\rangle\bar{f}_{c}\,\mathrm{d}x. \tag{B.2}$$

Hence we have

$$\begin{aligned}&\lim_{\varepsilon\to0}\frac{1}{2}\int_\Omega \langle\nabla\tilde{\xi}^\varepsilon, D_\varepsilon\nabla\tilde{\xi}^\varepsilon\rangle f^\varepsilon\,\mathrm{d}x\\ &=\frac{1}{2}\int_\Omega\langle\nabla\tilde{\xi},(\overline{D}+\overline{G})\nabla\tilde{\xi}\rangle f\,\mathrm{d}x\\ &\quad+\frac{1}{2}\int_\Omega\langle\nabla\tilde{\xi},(\overline{D}+\overline{G})\nabla\tilde{\xi}\rangle(\bar{f}_c-f)\,\mathrm{d}x+\lim_{\varepsilon\to0}\frac{1}{2}\int_\Omega \langle\nabla\tilde{\xi}^\varepsilon, D_\varepsilon\nabla\tilde{\xi}^\varepsilon\rangle (f_c-\bar{f}_c)\,\mathrm{d}x\end{aligned} \tag{B.3}$$

$$+\lim_{\varepsilon\to0}\frac{1}{2}\int_\Omega \langle\nabla\tilde{\xi}^\varepsilon, D_\varepsilon\nabla\tilde{\xi}^\varepsilon\rangle (f-f_c)\,\mathrm{d}x+\lim_{\varepsilon\to0}\frac{1}{2}\int_\Omega \langle\nabla\tilde{\xi}^\varepsilon, D_\varepsilon\nabla\tilde{\xi}^\varepsilon\rangle (f^\varepsilon-f)\,\mathrm{d}x. \tag{B.4}$$

A final ingredient we need is that the sequence of functions $\langle\nabla\tilde{\xi}^\varepsilon, D_\varepsilon\nabla\tilde{\xi}^\varepsilon\rangle$ is *equi-integrable*: *for all* $\sigma>0$, *there exists a* $\delta>0$ *such that for any* $S\subset\Omega$ *with* $|S|\le\delta$, *then*

$$\int_S\langle\nabla\tilde{\xi}^\varepsilon, D_\varepsilon\nabla\tilde{\xi}^\varepsilon\rangle\le\sigma \quad \text{holds for all } \varepsilon>0. \tag{B.5}$$

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Once this is shown, we can then make use of Lusin and Egorov Theorems to claim that all the terms in (B.3) and (B.4) converge to zero as $\varepsilon \to 0$: up to arbitrarily small measures, f equals a continuous function f_c, and the convergence of f^ε to f is uniform. We recall again that f^ε and f are uniformly bounded functions.

We now show that the sequence of functions $\tilde{\xi}^\varepsilon$ can be constructed so as it satisfies (B.5). Without loss of generality, we replace $\tilde{\xi}$ by a continuous and piece-wise affine function – this can be achieved by an approximation using Galerkin or finite element method (given that $\tilde{\xi}$ is smooth). Then we have a partition of Ω into a collection of polyhedrons. For simplicity, we can further assume that these polyhedrons are the same as the C_j on each of which $\bar{f}_c$ is constant. Now we construct $\tilde{\xi}^\varepsilon$ according to the following procedure.

First, we define $A(x, y) = D(x, y) = \pi(x, y)B^{-1}(y)$. By the smooth assumption of π and B, we have that A is smooth in $y \in \mathbb{T}^n$ and $x \in C_j$.

Now, for $x \in C_j$, as $\nabla\tilde{\xi}$ is a constant vector $p_j \in \mathbb{R}^n$, the homogenized matrix $\overline{A}(x)$ in Theorem 4.2 is given by (4.8) and is repeated here for convenience.

$$\langle\overline{A}(x)p_j, p_j\rangle = \inf\left\{\int_{\mathbb{T}^n} \langle A\,(x, y)\,(p_j + \nabla v),\ (p_j + \nabla v)\rangle\ \mathrm{d}y, \quad v \in H^1(\mathbb{T}^n)\right\}.$$

The inf above is achieved by $v_j(y) = |p_j|\hat{w}_j(x, y)$ where $\hat{w}_j$ solves the following cell-problem:

$$\mathrm{div}_y\left(A(x, y)\nabla\hat{w}_j\right) = -\mathrm{div}_y\left(A(x, y)\frac{p_j}{|p_j|}\right), \quad \hat{w}_j(x, \cdot) \in H^1(\mathbb{T}^n), \quad \int_{\mathbb{T}^n} \hat{w}_j(x, y)\ \mathrm{d}y = 0.$$

The smoothness assumption on A implies that

$$\|\hat{w}_j(x, \cdot), \quad \nabla_y\hat{w}_j(x, \cdot), \quad \nabla_x\hat{w}_j(x, \cdot)\|_{L^\infty(\mathbb{T}^2)} \le C$$

for some constant C that does not depend on x and ε.

Next, let $0 < d_1 < d_2$ be two positive numbers. For each C_j, there exists a smooth subdomain C_j' of C_j such that $d_1\varepsilon \le \mathrm{dist}(\partial C_j', \partial C_j) \le d_2\varepsilon$. Then we define a cut-off function η_j^ε on C_j satisfying: (i) $0 \le \eta_j^\varepsilon \le 1$ on C_j; (ii) $\eta_j^\varepsilon = 1$ on C_j'; and (iii) $\eta_j^\varepsilon(x) \longrightarrow 0$ smoothly as $x \longrightarrow \partial C_j$ so that $\eta_j^\varepsilon \in C_0^\infty(C_j)$; (iv) $\|\varepsilon\nabla\eta_j^\varepsilon\|_{L^\infty(C_j)} \le C$ for an ε-independent constant C.

With the above, suppose $\tilde{\xi}(x) = \sum_j \left[\alpha_j + \langle p_j, x\rangle\right]\chi_{C_j}(x)$, where χ_{C_j} is the characteristic function of C_j. We then define

$$\tilde{\xi}^\varepsilon(x) = \sum_j \left[\alpha_j + \langle p_j, x\rangle + \varepsilon\eta_j^\varepsilon(x)|p_j|\hat{w}_j(x, \frac{x}{\varepsilon})\right]\chi_{C_j}(x).$$

Then we have,

$$\nabla\tilde{\xi}^\varepsilon(x) = \sum_j \left[p_j + \eta_j^\varepsilon(x)|p_j|\nabla_y\hat{w}_j(x, \frac{x}{\varepsilon}) + \varepsilon\eta_j^\varepsilon(x)|p_j|\nabla_x\hat{w}_j(x, \frac{x}{\varepsilon}) + \varepsilon\nabla\eta_j^\varepsilon(x)|p_j|\hat{w}_j(x, \frac{x}{\varepsilon})\right]\chi_{C_j}(x).$$

By the aforementioned estimates for $\hat{w}_j$ and η_j^ε, we can conclude that $|\nabla\tilde{\xi}^\varepsilon(x)| \le C|p_j|$ for $x \in C_j$ and hence

$$|\nabla\tilde{\xi}^\varepsilon(x)| \le C|\nabla\tilde{\xi}(x)| \quad \text{for all } x \in \Omega.$$

(Here we make use of the $L^\infty(\mathbb{T}^n)$ estimates for $\hat{w}_j$ but we could also resort to the weaker $L^2(\mathbb{T}^n)$ estimates.) Note that the above statement holds uniformly for all $\varepsilon \ll 1$. We can then conclude (B.5) as $\int_\Omega |\nabla\tilde{\xi}|^2\,\mathrm{d}x$ is finite.

The fact that $\left\{\tilde{\xi}^\varepsilon\right\}_{\varepsilon>0}$ is a recovery sequence for $\tilde{\xi}$ is due to the properties that $\tilde{\xi}^\varepsilon \longrightarrow \tilde{\xi}$ in $L^2(\Omega)$ and $\nabla\tilde{\xi}^\varepsilon$ differs from the "optimal" oscillatory functions $\left\{p_j + |p_j|\nabla_y\hat{w}_j(x, \frac{x}{\varepsilon})\right\}_j$ only on $\bigcup_j C_j \backslash C_j'$ which has

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vanishing measure as $\varepsilon \longrightarrow 0$. More precisely, we have

$$\lim_{\varepsilon\to 0}\int \langle A(x,\frac{x}{\varepsilon})\nabla\tilde{\xi}^{\varepsilon},\nabla\tilde{\xi}^{\varepsilon}\rangle\bar{f}_c\,\mathrm{d}x=\lim_{\varepsilon\to 0}\sum_j\int_{C_j}\langle A(x,\frac{x}{\varepsilon})\nabla\tilde{\xi}^{\varepsilon},\nabla\tilde{\xi}^{\varepsilon}\rangle\bar{f}_{c_j}\,\mathrm{d}x$$

$$=\sum_j\int_{C_j}\int_{\mathbb{T}^n}\Big\langle A(x,y)\big(p_j+|p_j|\nabla_y\hat{w}_j(x,y)\big),\big(p_j+|p_j|\nabla_y\hat{w}_j(x,y)\big)\Big\rangle\,\mathrm{d}y\,\bar{f}_{c_j}\,\mathrm{d}x$$

$$=\sum_j\int_{C_j}\langle\bar{A}(x)p_j,p_j\rangle\bar{f}_{c_j}\,\mathrm{d}x=\int\langle\bar{A}(x)\nabla\tilde{\xi},\nabla\tilde{\xi}\rangle\bar{f}_c\,\mathrm{d}x.$$

The above computation is classical in the theory of two-scale convergence – see [3, Prop. 1.14(i), and equations (2.10), (2.11)]. Note also that (B.1) and (B.2) hold as $\bar{f}_c$ is constant on the C_j's.

We can now conclude (4.15).

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