# MATH-59800AOT Fall 2022

**Course:** Introduction to Optimal Transport
**CRN:** 28036
**Time:** TuTh 9:00 am - 10:15 am, Aug 23, 2022 – Dec 09, 2022
**Location:** Math 215
**Instructional Modality:** Face to Face

**Instructor:** Yuan GAO
**Office hour:** TuTh 1:30pm-3:00pm (Math 736) or other ways via Email appointment
**Office Location:** Math 736
**Email:** gao662@purdue.edu
**Office Phone number:** 765-496-0056
**Zoom room**: Meeting ID: 435 398 3929 Passcode: gao366

**Description:**

Credit Hours: 3.00. This is a graduate-level course introducing the subject of optimal transport. The mathematical theory of optimal transport can be formulated in terms of probability measures on an underlying metric space; the problem of mapping or transporting one measure to another leads to the definition of a metric on the set of probability measures which reflects the geometry of the underlying space. This elegant theory has connections to many topics within analysis, probability, geometry, statistics and data science. The first part of the course will develop the basic theory of optimal transport including original Monge problem, the dual formulation, polar factorization, the Monge-Ampere equation, the Wasserstein metric and the Benamou-Brenier method. The rest of the course will cover application of optimal transport ideas to partial differential equations (PDE), convex analysis, stochastic algorithm and data science. The goal of the course is to introduce the fundamental mathematical ideas in the field and to give a survey of some recent applications to other related areas.

**Texts:**

1. **Main textbook**: Ambrosio, Luigi, Elia Brué, and Daniele Semola. Lectures on optimal transport. Springer, 2021. https://link.springer.com/book/10.1007/978-3-030-72162-6
2. Other references:
   - Santambrogio, Filippo. "Optimal transport for applied mathematicians." Birkäuser, NY 55, no. 58-63 (2015): 94. https://link.springer.com/book/10.1007/978-3-319-20828-2
   - Villani, Cédric. Topics in optimal transportation. Vol. 58. American Mathematical Soc., 2021.
   - Gabriel Peyré and Marco Cuturi (2019), “Computational Optimal Transport”, Foundations and Trends in Machine Learning: Vol. 11, No. 5-6, pp 355–607.
3. Notes taken in class or posted on Brightspace.

**Important prerequisites:**
An essential prerequisite of the course is a thorough understanding of the concepts of measure theory. Some PDE and probability background is recommended but not required.
**Two assignments will be posted in Brightspace.**

There may be deviations from the schedule below, depending on class progress.

**Approximated Schedule** (subject to changes)**:**
Chapter sections refer to the main textbook.
1. Week of Aug 23: § 1.1-1.2, 2.1;
2. Week of Aug 30: § 2.2-2.4;
3. Week of Sep 06: § 3.1-3.4;
4. Week of Sep 13: § 4.1-4.6;
5. Week of Sep 20: § 5.1-5.4, (6.1);
6. Week of Sep 27: Take-home HW1 on Sep 27, no class on Sep 27; § 7.1-7.2
7. Week of Oct 04: § 8.1-8.3;
8. Week of Oct 11: no class on Oct 11 (fall break); § 9.1-9.2,
9. Week of Oct 18: §9.3, 11.1-11.2;
10. Week of Oct 25: §11.3, (12);
11. Week of Nov 01: § 13.1, (15);
12. Week of Nov 08: § 16.1-16.3;
13. Week of Nov 15: § 17.1-17.2; Take-home HW2 (TBD)
14. Week of Nov 22: § 18.1-18.2, no class on Nov 24 (Thanks giving day)
15. Week of Nov 29: Topic reports
16. Week of Dec 06: Topic reports

No Final Exam

**Topic reports references:**
Topic 1: Existence of EDE solution (Textbook Lecture 14, [GL])
Topic 2: Benamou-Brenier formula and algorithm ([Santambrogio's book 6.1][Peyre-Cuturi's book 7.3])
Topic 3: Relation between optimal transport and optimal control ([Chen])
Topic 4: JKO gradient flow: scheme and convergence ([JKO])
Topic 5: Regularized optimal transport with entropic panelty ([BB][EE])
Topic 6: Machine learning application ([SOT1,2] + code)
Topic 7: Image matching application ([IM][Ma])
Topic 8: Empirical measure comparison ([SS])
Topic 9: Schrodinger problem and large deviation ([Co])
Topic 10: Matrix-valued optimal transport ([Bo])
Topic 11: Incompressible optimal map ([LP])
Topic 12: Back-and-forth computational method for optimal transport ([Mt][Mt2] + code)

**Topic references**

[JKO]: Richard Jordan, David Kinderlehrer, and Felix Otto. The variational formulation of the Fokker-Planck equation. SIAM J. Math. Anal., 29(1):1–17, 1998.
[GL]: Gao, Yuan, and Jian-Guo Liu. "A note on parametric Bayesian inference via gradient flows." Annals of Mathematical Sciences and Applications 5, no. 2 (2020): 261-282.
[BB]: J-D. Benamou, G. Carlier, M. Cuturi, L. Nenna, and G. Peyré. Iterative bregman projections for regularized transportation problems. SIAM Journal on Scientific Computing, *37*(2), A1111-A1138, 2015.
[EE]: Peyré, Gabriel. "Entropic Approximation of Wasserstein Gradient Flows." SIAM Journal on Imaging Sciences 8, no. 4 (January 2015): 2323–51.
[SOT1]: Rolet, Antoine, Marco Cuturi, and Gabriel Peyre. "Fast Dictionary Learning with a Smoothed Wasserstein Loss," n.d., 9.

[SOT2]: Alvarez-Melis, David, Tommi Jaakkola, and Stefanie Jegelka. "Structured optimal transport." In International Conference on Artificial Intelligence and Statistics, pp. 1771-1780. PMLR, 2018..
[IM]: Dupuis, Paul, Ulf Grenander, and Michael I. Miller. “Variational Problems on Flows of Diffeomorphisms for Image Matching.” Quarterly of Applied Mathematics 56, no. 3 (September 1, 1998): 587–600.
[Ma]: Maas, Jan, Martin Rumpf, Carola Schönlieb, and Stefan Simon. "A generalized model for optimal transport of images including dissipation and density modulation." ESAIM: Mathematical Modelling and Numerical Analysis 49, no. 6 (2015): 1745-1769.
[SS]: Brown, Louis, and Stefan Steinerberger. “On the Wasserstein Distance between Classical Sequences and the Lebesgue Measure.” Transactions of the American Mathematical Society 373, no. 12 (October 5, 2020): 8943–62.
[Co]: Backhoff, Julio, Giovanni Conforti, Ivan Gentil, and Christian Léonard. “The Mean Field Schrödinger Problem: Ergodic Behavior, Entropy Estimates and Functional Inequalities.” Probability Theory and Related Fields 178, no. 1–2 (October 2020): 475–530.
[Chen]: Chen, Yongxin, Tryphon T. Georgiou, and Michele Pavon. "On the relation between optimal transport and Schrödinger bridges: A stochastic control viewpoint." Journal of Optimization Theory and Applications 169, no. 2 (2016): 671-691.
[Bo]: Li, Bowen, and Jun Zou. “On a General Matrix Valued Unbalanced Optimal Transport and Its Fully Discretization: Dynamic Formulation and Convergence Framework.” ArXiv:2011.05845 [Cs, Math], November 11, 2020. http://arxiv.org/abs/2011.05845.
[LP]: Liu, Jian-Guo, Robert L. Pego, and Dejan Slepčev. “Least Action Principles for Incompressible Flows and Geodesics between Shapes.” Calculus of Variations and Partial Differential Equations 58, no. 5 (October 2019): 179. https://doi.org/10.1007/s00526-019-1636-7.
[Mt]: M. Jacobs, W. Lee, and F. Leger, The back-and-forth method for Wasserstein gradient flows ESAIM:COCV, February 2021.
[Mt2]: Jacobs, Matt, and Flavien Léger. "A fast approach to optimal transport: The back-and-forth method." Numerische Mathematik 146, no. 3 (2020): 513-544.

**Grading System:**
1. Class attendance and activity: 10%

2. Open-book homework: 30%+30%
• Two, open-book homework. Submit within 48hours after the assignment.
• To be taken in or close to the week indicated in the schedule.
• Working in groups is encouraged! Mathematics can be a wonderfully collaborative endeavor. However, please submit **individual work, in your own words.**

3. Final topic report 30%
• Choose one topic (among 12) and give a 25min presentation/lecture in the last two weeks.

•Missed course work is officially accommodated in the following three circumstances:
1. Illness or other extraordinary personal circumstance
2. Religious observance
3. Varsity athletic participation
Late work for any other reason will not be accepted.

This policy may change if Purdue switches to online teaching.

**Grades policy**
Students who get at least 97% of the total points in this course are guaranteed an A+,

93% an A
90% an A-
87% a B+
83% a B
80% a B-
77% a C+
73% a C
70% a C-
67% a D+
63% a D
and 60% a D-

Above is departmental policy for the grade cut-offs. For each of these grades, it's possible that at the end of the semester a lower percentage will be enough to get that grade.

**Expectations:**
• This is an advanced course with high expectations. Your submitted work should reflect your best effort. Solutions should be complete, legible, and easily understood. Complete sentences expressing well-developed ideas should be used whenever appropriate.
• The goal of the course is to not only learn those classical method in the textbook but also learn the way of thinking. These fundamental ideas will (hopefully) influence the way you think and the way you solve problems. Thus, our goal is to not only teach you the content outlined in the course synopsis, but to also more broadly impact the way you think about problems in your chosen discipline.
• During reading and working out HWs, it is highly encouraged to explain concept, hear it from others in some different angles, and to answer other people's doubts, which will solidify one's own understanding.

**Academic Integrity**

Individuals are encouraged to alert university officials to potential breaches of this value by either emailing integrity@purdue.edu or by calling 765-494- 8778. While information may be submitted anonymously, the more information is submitted the greater the opportunity for the university to investigate the concern. More details are available on our course Brightspace table of contents, under University Policies.

Direct copying from (or too close to) other works or online materials is not allowed. Any form of cheating in exams will automatically lead to an F grade.

All materials, including Projects and HWs posted on the Brightspace of this course subject to my copyright and cannot be bartered.

**Academic Guidance in the Event a Student is Quarantined/Isolated**
If you must quarantine or isolate at any point in time during the semester, contact the Protect Purdue Health Center at 765-496-4636. Please also reach out to me via email so that we can communicate

about how you can continue to learn remotely. Work with the Protect Purdue Health Center (PPHC) to get documentation and support, including access to an Academic Case Manager who can provide you with general guidelines/resources around communicating with your instructors, be available for academic support, and offer suggestions for how to be successful when learning remotely. Your Academic Case Manager can be reached at acmq@purdue.edu. Importantly, if you find yourself too sick for an extended period of time to progress in the course, notify your academic case manager and notify me via email or Brightspace. We will make arrangements based on your particular situation.

**Classroom Guidance Regarding Protect Purdue**

Please refer to the Protect Purdue Plan including the Protect Purdue Pledge, for campus policy and as such all members of the Purdue community must comply with the required health and safety guidelines.

Please refer to the latest Protect Purdue Pledge and Classroom Expectation below.

https://protect.purdue.edu/pledge/

https://www.purdue.edu/innovativelearning/download/sop-for-classrooms-instructional-lab-experiential-course_fall-2022/?wpdmdl=4557&refresh=62e7f410521841659368464

**Lack of compliance**

Students who are not engaging in behaviors established in the standard operating procedures (e.g., properly wearing a mask when required) will be asked to comply and offered any assistance they need in order to comply. If non-compliance continues, possible results include instructors asking students to leave the class, potentially followed by instructors dismissing the whole class. Students who do not comply with the required health and Protect Purdue Pledge behaviors are violating the University Code of Conduct and will be reported to the Dean of Students Office, with sanctions ranging from educational requirements to dismissal from the university. For additional guidance, please see the Dean of Students guidance on Managing Classroom Behavior and Expectations.

**Student rights**

Any student who has substantial reason to believe that another person in the room is threatening class safety by not wearing a face covering or following other safety guidelines for public health considerations may leave the class without consequence. The student is encouraged to report the observed behavior to the course instructor or to the Office of Student Rights and Responsibilities (OSRR), as well as discuss next steps with the instructor.

**Accommodations for Students with Disabilities and Academic Adjustment:**

Purdue University strives to make learning experiences accessible to all participants. If you anticipate or experience physical or academic barriers based on disability, you are welcome to let me know so that we can discuss options. You are also encouraged to contact the Disability Resource Center at: drc@purdue.edu or by phone at 765-494-1247.

If you have been certified by the Disability Resource Center (DRC) as eligible for accommodations, you should contact your instructor to discuss your accommodations as soon as possible. Here are instructions for sending your Course Accessibility Letter to your instructor:https://www.purdue.edu/drc/students/course-accessibility-letter.php

**Nondiscrimination Statement:**

This class, as part of Purdue University's educational endeavor, is committed to maintaining a community which recognizes and values the inherent worth and dignity of every person; fosters tolerance, sensitivity, understanding, and mutual respect among its members; and encourages each individual to strive to reach his or her own potential.

**Mental Health/Wellness Statement**

Help-seeking is a life skill rather than an indication of weakness.

If you find yourself beginning to feel some stress, anxiety and/or feeling slightly overwhelmed, try WellTrack.

If you need support and information about options and resources, please contact or see the Office of the Dean of Students. Call 765-494-1747. Hours of operation are M-F, 8 am- 5 pm.

If you find yourself struggling to find a healthy balance between academics, social life, stress, etc. sign up for free one- on-one virtual or in-person sessions with a Purdue Wellness Coach at RecWell. Student coaches can help you navigate through barriers and challenges toward your goals throughout the semester. Sign up is completely free and can be done on BoilerConnect. If you have any questions, please contact Purdue Wellness at evans240@purdue.edu.

If you're struggling and need mental health services: Purdue University is committed to advancing the mental health and well-being of its students. If you or someone you know is feeling overwhelmed, depressed, and/or in need of mental health support, services are available. For help, such individuals should contact Counseling and Psychological Services (CAPS) at 765-494-6995 during and after hours, on weekends and holidays, or by going to the CAPS office on the second floor of the Purdue University Student Health Center (PUSH) during business hours.

**Emergency Preparation**

In the event of a major campus emergency, course requirements, deadlines and grading percentages are subject to changes that may be necessitated by a revised semester calendar or other circumstances beyond the instructor's control. Relevant changes to this course will be posted onto the course website or can be obtained by contacting the instructors or TAs via email or phone. You are expected to read your @purdue.edu email on a frequent basis. A link to Purdue's Emergency Preparedness resources (also located on the Brightspace shell under University Policies) https://www.purdue.edu/ehps/emergency_preparedness/