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Calculus of Variations

Existence and uniqueness of bounded stable solutions to the Peierls–Nabarro model for curved dislocations

Hongjie Dong[1] **· Yuan Gao**[2]

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Abstract

We study the well-posedness of the vector-field Peierls–Nabarro model for curved dislocations with a double well potential and a bi-states limit at far field. Using the Dirichlet to Neumann map, the 3D Peierls–Nabarro model is reduced to a nonlocal scalar Ginzburg–Landau equation. We derive an integral formulation of the nonlocal operator, whose kernel is anisotropic and positive when Poisson's ratio $\nu \in (-\frac{1}{2}, \frac{1}{3})$. We then prove that any bounded stable solution to this nonlocal scalar Ginzburg–Landau equation has a 1D profile, which corresponds to the PDE version of flatness result for minimal surfaces with anisotropic nonlocal perimeter. Based on this, we finally obtain that steady states to the nonlocal scalar equation, as well as the original Peierls–Nabarro model, can be characterized as a one-parameter family of straight dislocation solutions to a rescaled 1D Ginzburg–Landau equation with the half Laplacian.

Mathematics Subject Classification 35A02 · 35J50 · 35Q74 · 35R11 · 35J60

1 Introduction

Materials defects such as dislocations are important structures in crystalline materials and play essential roles in the study of plastic and mechanical behaviors of materials. Along the dislocation line, there is a small region (called the dislocation core region) of heavily distorted atomistic structures with shear displacement jump across a slip plane, denoted by

$$\Gamma := \{(x_1, x_2, x_3) : \ x_3 = 0\}.$$

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✉ Yuan Gao
yuangao@math.duke.edu

Hongjie Dong
hongjie_dong@brown.edu

[1] Division of Applied Mathematics, Brown University, 182 George Street, Providence, RI 02912, USA

[2] Department of Mathematics, Duke University, Durham, NC 27708, USA

The classical dislocation theory [14] regards the dislocation core as a singular point so that the solution can be solved explicitly based on the linear elasticity theory, which, however, is not able to unveil detailed core structure of dislocations. Instead, the Peierls–Nabarro (PN) model introduced by Peierls and Nabarro [15,17] is a multiscale continuum model for displacement $\mathbf{u} = (u_1, u_2, u_3)$ that incorporates the atomistic effect by introducing a nonlinear potential describing the atomistic misfit interaction across the slip plane Γ of the dislocation. More precisely, assume two elastic continua $x_3 > 0$ and $x_3 < 0$ are connected by a nonlinear atomistic potential γ depending on shear displacement jump

$$([u_1], [u_2]) := (u_1(x_1, x_2, 0^+) - u_1(x_1, x_2, 0^-),\ u_2(x_1, x_2, 0^+) - u_2(x_1, x_2, 0^-))$$

across the slip plane Γ. Although the total jump increment is determined by the magnitude of Burgers vector, the true spread of the jump increment $([u_1], [u_2])$ is determined globally by the whole system, particularly for curved dislocation with variant orientations. Given the magnitude of the Burgers vector, which means that given the bi-states boundary conditions at far field (see (2.3)), the problem turns out to be a minimization problem of the total energy

$$E(\mathbf{u}) := E_{\text{els}}(\mathbf{u}) + E_{\text{mis}}(\mathbf{u});$$

see the detailed definitions of the elastic energy $E_{\text{els}}(\mathbf{u})$ and the misfit energy $E_{\text{mis}}(\mathbf{u})$ in Sect. 2.1. The resulting Euler–Lagrange equations for the vector-field $\mathbf{u}$ is a Láme system with a nonlinear boundary condition; see (2.6).

For a straight dislocation with uniform displacement in the x_2 direction, the 2D Láme system with the nonlinear boundary condition can be reduced to a nonlocal equation (also known as nonlocal Ginzburg–Landau equation with double-well potential γ)

$$(-\Delta)^{\frac{1}{2}}\tilde{u}(x_1) = -\gamma'(\tilde{u}(x_1)), \quad x_1 \in \mathbb{R} \tag{1.1}$$

with the bi-states at far field $\tilde{u}(\pm\infty) = \pm 1$. For a special sinusoidal misfit potential reflecting phenomenologically lattice periodicity $\gamma(\tilde{u}) = \frac{1}{\pi^2}(1 + \cos(\pi\tilde{u}))$, with certain physical constants for computational simplicity, the solution can be solved explicitly [14,20] with shear displacement

$$\tilde{u}(x_1) = \frac{2}{\pi}\arctan(x_1) \sim \pm 1 - \frac{2}{\pi x_1} \quad \text{as } x_1 \to \pm\infty.$$

Equation (1.1), as well as the corresponding scalar displacement $\tilde{u}(x_1, x_2)$, as the harmonic extension of $\tilde{u}(x_1, 0)$, is well studied recently at a rigorous mathematical level. For a general misfit potential γ with $C^{2,\alpha}$ regularity [5], Cabré and Solà-Morales established the existence (and the uniqueness up to translations) of monotonic solutions with the sharp decay rate $\frac{1}{x_1}$ for the bistable profile. They also proved that the bistable profile is a local minimizer with respect to perturbations with compact support for the total energy

$$E(\tilde{u}) = \frac{1}{2}\int |\nabla\tilde{u}|^2\,\mathrm{d}x_1\,\mathrm{d}x_2 + \int_\Gamma \gamma(\tilde{u})\,\mathrm{d}x_1$$

of the scalar model using the harmonic extension. Without using the harmonic extension, Palatucci, Savin, and Valdinoci directly worked on the nonlocal equation $(-\Delta)^{\frac{1}{2}}\tilde{u}|_\Gamma = -\gamma'(\tilde{u})$ on Γ and improved the global minimizer result by proving quantitative growth estimates of the total energy [16]. In [12], the authors established rigorously the connection between the vector-field 2D Láme system and the reduced equation (1.1) at both the equation and energy level by considering a perturbed energy. Besides, for the De Giorgi-type hyperplane conjecture for stable solutions to (1.1) (also known as De Giorgi-type hyperplane conjecture for the Laplace equation with nonlinear boundary reaction), it was proved in [5] that in 2D, bounded stable solutions have 1D profiles. For a general nonlocal Ginzburg–Landau equation

$$(-\Delta)^s \tilde{u} = -\gamma'(\tilde{u}), \quad x \in \mathbb{R}^d,$$

we refer the readers to some recent results for $d = 2, 0 < s < 1$ by Cabré and Sire [2]; for $d = 3, s = \frac{1}{2}$ by Cabré and Cinti [3]; for $d = 3$, $\frac{1}{2} < s < 1$ by Cabré and Cinti [4] and for $d = 4, s = \frac{1}{2}$ by Figalli and Serra [10]; and related energy estimates for $0 < s < 1$ by Gui and Li [13] and flatness results for $0 < s < \frac{1}{2}$ by Dipierro, Serra, and Valdinoci [7,8] and for $\frac{1}{2} \leq s < 1$ by Savin [18].

However, to our best knowledge, so far there is no result for the 3D vector-field system (2.6), which cannot be treated as an analogue scalar model above. In fact, the vector-field displacement is essential to determine the long-range elastic interaction associated with dislocations and partial separation within the dislocation core. We are especially interested in the curved dislocation [19], which is the most common case, and their properties are anisotropic in space and depend on the orientations of dislocations. The main goal of this paper is to study in which cases, the steady state (equilibrium) of the PN model (2.6) has to be a straight dislocation. This also establishes the foundation of further researches on the dynamics and long-time behaviors of curved dislocations.

There are in general two strategies to study the solutions to the full system (2.6). One is to study the local vector-field system with nonlinear Neumann boundary conditions. However, the challenges come from the lack of maximum principle and the lack of the compactness in unbounded domain. The other strategy is to reduce the 3D full system to a nonlocal 2D problem using Dirichlet to Neumann map, which, in the curved dislocation case, is still a coupled nonlocal system; see (2.8). Under the assumption that the misfit potential γ depends only on the shear jump displacement $[u_1]$, we will further reduce it to a scalar nonlocal equation (see (2.15)) and study the resulting nonlocal operator, which shows anisotropic property in different directions. The kernel of the new nonlocal operator is still homogeneous but anisotropic; see Propositions 3.2 and 3.3. Especially, the kernel remains positive only for Poisson's ratio ν in the range $(-\frac{1}{2}, \frac{1}{3})$.

With a positive anisotropic kernel, a natural question is the existence and uniqueness of solutions to the nonlocal equation. Since the straight dislocation is a special solution to the full system, we are particularly interested in the characterization of the solutions, i.e., if the misfit potential depends only on $[u_1]$, whether the straight solution is the only stable solution to the full system (2.6). We will follow the idea in [6], which proves quantitative flatness estimates for the stable sets for nonlocal perimeters (see also [7,10,13,18] for PDE version with fractional Laplacian), to first show that any bounded stable solution to (2.15) has a 1D profile; see Theorem 4.6. As a consequence, all the solutions to (2.15) as well as (2.6) can be characterized as a rotation of straight dislocation. This is analogue to the flatness result for the isotropic case with the half Laplacian. However, for the general case when the misfit potential depends both on $[u_1]$ and $[u_2]$, the characterization of solutions to the coupled nonlocal Ginzburg–Landau system (2.8) remains open.

The paper will be organized as follows. In Sect. 2, we propose the governing equations for the full vector-field system and then reduce it to a nonlocal equation (2.15) by the Dirichlet to Neumann map. In Sect. 3, we derive the integral formulation of the new nonlocal operator and study the positivity of the resulting anisotropic kernel. In Sect. 4, we prove that any bounded stable solution to the reduced nonlocal equation (2.15) has a 1D monotone profile and is given by a rotation of straight dislocation. The derivation of the Euler–Lagrange equation and the Dirichlet to Neumann map will be given in Appendices A and B, respectively.

2 Full system and reduced nonlocal system by the Dirichlet to Neumann map

In this section, we will first derive the Euler–Lagrange equation for the PN model, which is a minimization problem of the total energy consisting of the elastic energy and the misfit energy induced by a dislocation; see Sect. 2.1. Then we will derive the reduced nonlocal systems/equation by the Dirichlet to Neumann map in Sect. 2.2.

2.1 Vector-field full system with nonlinear boundary condition

In the PN model, the two half spaces separated by the slip plane $\Gamma = \{(x_1, x_2, x_3); x_3 = 0\}$ of the dislocation are assumed to be linear elastic continua, and the two half spaces are connected by a nonlinear potential energy across the slip plane that incorporates atomistic interactions. Let us first clarify the total energy, which is indeed infinite in $\mathbb{R}^3$ due to the presence of a dislocation [12], and then derive the Euler–Lagrange equation by regarding the solution as a local minimizer of the total energy.

Let $\mathbf{u} = (u_1, u_2, u_3)$ be the displacement vector. The total energy $E(\mathbf{u})$ of the whole system is

$$E(\mathbf{u}) := E_{\text{els}}(\mathbf{u}) + E_{\text{mis}}(\mathbf{u}). \tag{2.1}$$

Let $G > 0$ be the shear modulus and $\nu \in [-1, \frac{1}{2}]$ be Poisson's ratio. The first term in the total energy in Eq. (2.1) is the elastic energy in the two half spaces defined as

$$E_{\text{els}} = \int_{\mathbb{R}^3\setminus\Gamma} \frac{1}{2}\varepsilon : \sigma \,\mathrm{d}x = \int_{\mathbb{R}^3\setminus\Gamma} \frac{1}{2}\varepsilon_{ij}\sigma_{ij} \,\mathrm{d}x,$$

where ε is the strain tensor

$$\varepsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j) \quad \text{for } i, j = 1, 2, 3, \quad \partial_i := \frac{\partial}{\partial x_i},$$

and σ is the stress tensor

$$\sigma_{ij} = 2G\varepsilon_{ij} + \frac{2\nu G}{1 - 2\nu}\varepsilon_{kk}\delta_{ij} \quad \text{for } i, j = 1, 2, 3.$$

Here $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. We also used the Einstein summation convention that

$$\varepsilon_{kk} = \sum_{k=1}^{3} \varepsilon_{kk} \quad \text{and} \quad \sigma_{ij}\varepsilon_{ij} = \sum_{i,j=1}^{3} \sigma_{ij}\varepsilon_{ij}.$$

The second term in the total energy in Eq. (2.1) is the misfit energy across the slip plane due to nonlinear atomistic interactions

$$E_{\text{mis}}(\mathbf{u}) := \int_{\Gamma} \gamma(u_1^+ - u_1^-, u_2^+ - u_2^-) \,\mathrm{d}\Gamma = \int_{\Gamma} W(u_1^+, u_2^+) \,\mathrm{d}\Gamma,$$

where $u_i^{\pm} := u_i(x_1, x_2, 0^{\pm})$ for $i = 1, 2$. For the analysis of the PN model for an edge dislocation in this paper, we assume that the nonlinear potential $W \in C_b^{2,\alpha}(\mathbb{R}^2; \mathbb{R})$ for some $\alpha \in (0,1)$, i.e., W is twice-continuously differentiable with bounded derivatives up to second order and D^2W is α-Hölder continuous. In practice, W will be a periodic potential indicating the periodic lattice structure of the materials with several minimums, for instance $W(v_1, v_2) = \cos v_1 + \sin v_2$, and will be specified later.

The equilibrium structure of a general curved dislocation is obtained by minimizing the total energy in Eq. (2.1) subject to the boundary condition at the slip plane

$$\begin{aligned} u_1^+(x_1,x_2) &= -u_1^-(x_1,x_2),\\ u_2^+(x_1,x_2) &= -u_2^-(x_1,x_2),\\ u_3^+(x_1,x_2) &= -u_3^-(x_1,x_2). \end{aligned} \tag{2.2}$$

To focus on nontrivial solutions indicating the presence of a curved dislocation, we consider the following bi-states far field boundary condition for u_1,

$$u_1^+(\pm\infty, x_2, 0^+) = \pm 1 \quad \text{for any } x_2 \in \mathbb{R}, \tag{2.3}$$

where we chose certain magnitude of the Burgers vector for simplicity.

However, due to the slow decay rate of the strain tensor ε, we have the same issue with straight dislocation as in [5,12,16], i.e., the elastic energy is infinite. Whenever the total energy is infinite, we define the energy minimizer in the perturbed sense with respect to a perturbation with compact support. To be precise, we define the perturbed elastic energy of $\mathbf{u}$ with respect to any perturbation fields $\boldsymbol{\varphi} \in C^\infty(\mathbb{R}^3\backslash\Gamma;\mathbb{R}^3)$ and $\boldsymbol{\varphi}$ has compact support in some $B_R \subset \mathbb{R}^3$ as

$$\begin{aligned} \hat{E}_{\text{els}}(\boldsymbol{\varphi}|\mathbf{u}) :=& \int_{\mathbb{R}^3\backslash\Gamma} \frac{1}{2}(\varepsilon_u+\varepsilon_\varphi):(\sigma_u+\sigma_\varphi) - \frac{1}{2}\varepsilon_u:\sigma_u \,\mathrm{d}x\\ =& \int_{\mathbb{R}^3\backslash\Gamma} \frac{1}{2}[(\varepsilon_\varphi)_{ij}(\sigma_\varphi)_{ij} + (\varepsilon_\varphi)_{ij}(\sigma_u)_{ij} + (\varepsilon_u)_{ij}(\sigma_\varphi)_{ij}] \,\mathrm{d}x\\ =& E_{\text{els}}(\boldsymbol{\varphi}) + \mathcal{C}_{\text{els}}(\mathbf{u},\boldsymbol{\varphi}), \end{aligned}$$

where the cross term

$$\mathcal{C}_{\text{els}}(\mathbf{u},\boldsymbol{\varphi}) := \int_{\mathbb{R}^3\backslash\Gamma} \frac{1}{2}(\varepsilon_\varphi:\sigma_u + \varepsilon_u:\sigma_\varphi)\,\mathrm{d}x = \int_{\mathbb{R}^3\backslash\Gamma} \frac{1}{2}[(\varepsilon_\varphi)_{ij}(\sigma_u)_{ij} + (\varepsilon_u)_{ij}(\sigma_\varphi)_{ij}]\,\mathrm{d}x,$$

and ε_u, σ_u and ε_φ, σ_φ are the strain and stress tensors corresponding to $\mathbf{u}$ and $\boldsymbol{\varphi}$, respectively. Then the perturbed total energy is defined as

$$\hat{E}(\boldsymbol{\varphi}|\mathbf{u}) := \hat{E}_{\text{els}}(\boldsymbol{\varphi}|\mathbf{u}) + \int_\Gamma W(u_1+\varphi_1) - W(u_1)\,\mathrm{d}x \tag{2.4}$$

and the energy minimizer is defined as $\mathbf{u}$ such that $\hat{E}(\boldsymbol{\varphi}|\mathbf{u}) \geq 0$ for any $\boldsymbol{\varphi}$ with compact support.

Remark 1 Since $\mathbf{u}$ and $\mathbf{u}+\boldsymbol{\varphi}$ coincide outside B_R, we always know that $\hat{E}_{\text{els}}(\boldsymbol{\varphi}|\mathbf{u})$ is equivalent to the local perturbed elastic energy

$$\hat{E}_{\text{els}}(\boldsymbol{\varphi}|\mathbf{u};B_R) := \int_{B_R\backslash\Gamma} \frac{1}{2}(\varepsilon_u+\varepsilon_\varphi):(\sigma_u+\sigma_\varphi) - \frac{1}{2}\varepsilon_u:\sigma_u\,\mathrm{d}x = E_{\text{els}}(\mathbf{u}+\boldsymbol{\varphi};B_R) - E_{\text{els}}(\mathbf{u};B_R)$$

and $\hat{E}(\boldsymbol{\varphi}|\mathbf{u})$ is equivalent to

$$\hat{E}(\boldsymbol{\varphi}|\mathbf{u};B_R) := \hat{E}_{\text{els}}(\boldsymbol{\varphi}|\mathbf{u};B_R) + \int_{B_R\cap\Gamma} W(u_1+\varphi_1) - W(u_1)\,\mathrm{d}x = E(\mathbf{u}+\boldsymbol{\varphi};B_R) - E(\mathbf{u};B_R).$$

Therefore, we will follow the convention in [5,16] that $\mathbf{u}$ is referred to as a local minimizer. In the remaining part of the paper, whenever we consider the equivalence of two infinite energies, it is understood in the perturbed sense [12] or equivalently, in the local sense in any balls B_R.

Definition 1 We call the function $\mathbf{u}$ a local minimizer of total energy E if it satisfies

$$E(\mathbf{u}+\boldsymbol{\varphi}; B_R) - E(\mathbf{u}; B_R) \geq 0$$

for any perturbation $\boldsymbol{\varphi} \in C^\infty(\mathbb{R}^3\backslash\Gamma; \mathbb{R}^3)$ supported in some B_R satisfying

$$\begin{aligned} \varphi_1^+(x_1,x_2) &= -\varphi_1^-(x_1,x_2), \\ \varphi_2^+(x_1,x_2) &= -\varphi_2^-(x_1,x_2), \\ \varphi_3^+(x_1,x_2) &= \varphi_3^-(x_1,x_2). \end{aligned} \tag{2.5}$$

We have the following lemma for the Euler–Lagrange equation with respect to the total energy $E(\mathbf{u})$, which gives the governing equations for the vector-field full system. The proof of this lemma will be included in Appendix A for completeness.

Lemma 2.1 *Assume that* $\mathbf{u} \in C^2(\mathbb{R}^3\backslash\Gamma; \mathbb{R}^3)$ *satisfying the boundary conditions* (2.2) *and* (2.3) *is a local minimizer of the total energy E in the sense of Definition* 1*. Then* $\mathbf{u}$ *satisfies the Euler–Lagrange equation*

$$\begin{aligned} &(1-2\nu)\Delta\mathbf{u} + \nabla(\nabla\cdot\mathbf{u}) = 0 \quad \text{in } \mathbb{R}^3\backslash\Gamma, \\ &\sigma_{13}^+ + \sigma_{13}^- = \partial_1 W(u_1^+, u_2^+) \quad \text{on } \Gamma, \\ &\sigma_{23}^+ + \sigma_{23}^- = \partial_2 W(u_1^+, u_2^+) \quad \text{on } \Gamma, \\ &\sigma_{33}^+ = \sigma_{33}^- \quad \text{on } \Gamma. \end{aligned} \tag{2.6}$$

2.2 Dirichlet to Neumann map and the reduced nonlocal problem

In this section, we first take the strategy which reduces the 3D vector-field full system to a nonlocal system in $\mathbb{R}^2$ using the Dirichlet to Neumann map. Then we will focus on the case that misfit potential W depends only on the shear jump of the first component of the displacement field, which allows us to further reduce the problem to a scalar nonlocal Ginzburg–Landau equation in $\mathbb{R}$.

2.2.1 Reduction of the 3D full system to a nonlocal 2D system

First we give the following Dirichlet to Neumann map such that the vector-field displacement $\mathbf{u}$ can be expressed by the Dirichlet values of u_1, u_2 on Γ. The proof of this lemma is standard and will be given in Appendix B. Define the homogeneous Sobolev space as

$$\dot{H}^s(\mathbb{R}^2) := \Big\{u;\ \int_{\mathbb{R}^2} |\xi|^{2s}|\hat{u}(\xi)|^2\,\mathrm{d}\xi < \infty\Big\}.$$

Lemma 2.2 (Dirichlet to Neumann map) *Assume that* $\mathbf{u}$ *is the solution to* (2.6) *such that the Dirichlet value of u_1, u_2 on Γ are in $\dot{H}^s(\mathbb{R}^2)$ for some $s \geq \frac{1}{2}$. Then solution* $\mathbf{u}$ *can be* *determined uniquely by $u_1|_\Gamma, u_2|_\Gamma$. Particularly, $\sigma_{13}(x_1,x_2,0^+)$ and $\sigma_{23}(x_1,x_2,0^+)$ can be expressed by their Fourier transforms as*

$$\begin{pmatrix}\hat\sigma_{13}(k)\\ \hat\sigma_{23}(k)\end{pmatrix} = -A\begin{pmatrix}\hat u_1(k)\\ \hat u_2(k)\end{pmatrix} := 2G\begin{pmatrix} -\left(\frac{k_2^2}{|k|}+\frac{1}{(1-\nu)}\frac{k_1^2}{|k|}\right)\hat u_1(k) - \frac{\nu}{(1-\nu)}\frac{k_1k_2}{|k|}\hat u_2(k)\\ -\frac{\nu}{(1-\nu)}\frac{k_1k_2}{|k|}\hat u_1(k) - \left(\frac{k_1^2}{|k|}+\frac{1}{(1-\nu)}\frac{k_2^2}{|k|}\right)\hat u_2(k)\end{pmatrix}, \tag{2.7}$$

where $k=(k_1,k_2)$ is the frequency vector, $|k|=\sqrt{k_1^2+k_2^2}$, and $\nu\in[-1,\frac12]$ is Poisson's ratio.

Without loss of generality, from now on, we set shear modulus G to be $\frac12$ and use the notation $u_1=u_1^+(x_1,x_2,0^+),\ u_2=u_2^+(x_1,x_2,0^+),\ (x_1,x_2)\in\Gamma$, for simplicity.

From the Dirichlet to Neumann map in Lemma 2.2, the nonlinearity is decoupled and we obtain a 2D nonlocal system

$$\begin{pmatrix}-\sigma_{13}^+ - \sigma_{13}^-\\ -\sigma_{23}^+ - \sigma_{23}^-\end{pmatrix} =: \mathcal{A}\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \begin{pmatrix}-\partial_1 W(u_1,u_2)\\ -\partial_2 W(u_1,u_2)\end{pmatrix} \quad \text{on } \Gamma, \tag{2.8}$$

where the nonlocal operator $\mathcal{A}$ is expressed below in (2.9). For straight dislocations, we refer to [12] for details about the equivalence between the full system and reduced 1D equation in terms of both solutions and energies. Below, we formally derive the equivalence of the energies for the full system (2.6) and the reduced problem (2.8).

Recall that

$$A = \begin{pmatrix}\frac{k_2^2}{|k|}+\frac{1}{(1-\nu)}\frac{k_1^2}{|k|} & \frac{\nu}{(1-\nu)}\frac{k_1k_2}{|k|}\\ \frac{\nu}{(1-\nu)}\frac{k_1k_2}{|k|} & \frac{k_1^2}{|k|}+\frac{1}{(1-\nu)}\frac{k_2^2}{|k|}\end{pmatrix} = \begin{pmatrix}|k| & 0\\ 0 & |k|\end{pmatrix} + \frac{\nu}{1-\nu}\begin{pmatrix}\frac{k_1^2}{|k|} & \frac{k_1k_2}{|k|}\\ \frac{k_1k_2}{|k|} & \frac{k_2^2}{|k|}\end{pmatrix},$$

which is positive definite for Poisson's ratio $\nu\in(-1,\frac12)$. For $x=(x_1,x_2)$, $x'=(x_1',x_2')$, recall the Riesz potential in 2D is

$$I_\alpha f(\mathbf{x}) := c\int_{\mathbb{R}^2}|x-x'|^{-2+\alpha}f(x')\,\mathrm{d}x', \quad 0<\alpha<2$$

with the Fourier symbol $|k|^{-\alpha}$. Thus for $r:=\sqrt{x_1^2+x_2^2}=|x|$,

$$\mathcal{F}^{-1}\left(\frac{k_ik_j}{|k|}\hat f\right) = \partial_{ij}\frac{1}{r} * f, \quad i,j=1,2,$$

where $\mathcal{F}$ means the Fourier transformation. We can rewrite $\mathcal{A}$ as

$$\mathcal{A}\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \int_{\mathbb{R}^2} G(x-x')\begin{pmatrix}u_1(x)-u_1(x')\\ u_2(x)-u_2(x')\end{pmatrix}\mathrm{d}x', \tag{2.9}$$

where

$$G(x) := \frac{1}{r^3}\left[\frac{1-2\nu}{1-\nu}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{3\nu}{1-\nu}\begin{pmatrix}\frac{x_1^2}{r^2} & \frac{x_1x_2}{r^2}\\ \frac{x_1x_2}{r^2} & \frac{x_2^2}{r^2}\end{pmatrix}\right] = \frac{1}{|x|^3}\left(\frac{1-2\nu}{1-\nu}I + \frac{3\nu}{1-\nu}\frac{x}{|x|}\otimes\frac{x}{|x|}\right).$$

For $\nu\in(-1,\frac12)$, since A is positive defined, from Plancherel's equality, we have

$$c_2\|(u_1,u_2)\|^2_{\dot H^{\frac12}(\Gamma)} \le \int_{\mathbb{R}^2}(u_1,u_2)^T\mathcal{A}\begin{pmatrix}u_1\\ u_2\end{pmatrix}\mathrm{d}x \le C_2\|(u_1,u_2)\|^2_{\dot H^{\frac12}(\Gamma)}.$$

Rigorously, the inequality should be understood in perturbed sense; see [12]. Denote the reduced energy on Γ as

$$E_\Gamma := \frac{1}{2}\int_{\mathbb{R}^2} (u_1, u_2)^T \mathcal{A} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mathrm{d}x + \int_{\mathbb{R}^2} W(u_1, u_2)\, \mathrm{d}x. \tag{2.10}$$

Similar to (2.4), we define the perturbed elastic energy of $\mathbf{u}$ on Γ with respect to the perturbation $\boldsymbol{\varphi} \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$ as

$$\hat{E}_{\Gamma_e}(\boldsymbol{\varphi}|\mathbf{u}) := \frac{1}{2}\int_\Gamma (u_1 + \varphi_1, u_2 + \varphi_2)^T \mathcal{A} \begin{pmatrix} u_1 + \varphi_1 \\ u_2 + \varphi_2 \end{pmatrix} - (u_1, u_2)^T \mathcal{A} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mathrm{d}x$$

and the perturbed total energy on Γ as

$$\hat{E}_\Gamma(\boldsymbol{\varphi}|\mathbf{u}) := \hat{E}_{\Gamma_e}(\boldsymbol{\varphi}|\mathbf{u}) + \int_\Gamma W(u_1 + \varphi_1, u_2 + \varphi_2) - W(u_1, u_2)\, \mathrm{d}x.$$

One can check the straight solution uniform in x_2, i.e., $u_1(x_1, x_2) = \phi(x_1),\ u_2(x_1, x_2) = 0$ is a solution to (2.8), where $\phi(x_1)$ is the solution to the 1D problem

$$\begin{aligned} &(-\Delta)^{\frac{1}{2}}\phi(x_1) = -(1-\nu)W'(\phi(x_1)), \quad x_1 \in \mathbb{R}, \\ &\lim_{x_1 \to \pm\infty} \phi(x_1) = \pm 1. \end{aligned} \tag{2.11}$$

We refer to [5,16] for the existence and uniqueness to (2.11), which also proved that ϕ is bounded, increasing from -1 to 1, and a local minimizer of the corresponding 1D energy. See also [11] in which the authors proved that ϕ is the unique equilibrium of the corresponding 1D nonlocal dynamics Ginzburg–Landau equation. However, there might be other solutions to (2.6). In the next section, we will further reduce the nonlocal system to a 1D nonlocal equation for the case of a potential depending only on $[u_1]$ and characterize bounded stable solutions.

2.2.2 Reduction of the 2D nonlocal system to a 1D equation

If the misfit potential W depends only on one component of displacement jump, i.e., $W(u_1, u_2) = W(u_1)$, we can reduce the 2D system further to a scalar equation. Let us first clarify the assumption on the double well/periodic potential W:

$$\begin{aligned} &W \in C_b^{2,\alpha}(\mathbb{R}; \mathbb{R}), \\ &W(x) > W(\pm 1), \quad x \in (-1, 1), \\ &W''(\pm 1) > 0. \end{aligned} \tag{2.12}$$

In the case when $W(u_1, u_2) = W(u_1)$, from (2.7), we represent $\hat{u}_2$ by $\hat{u}_1$, i.e.,

$$\frac{\nu}{1-\nu}\frac{k_1 k_2}{|k|}\hat{u}_1(k) + \left(\frac{k_1^2}{|k|} + \frac{1}{1-\nu}\frac{k_2^2}{|k|}\right)\hat{u}_2(k) = 0, \tag{2.13}$$

which is equivalent to

$$\hat{u}_2(k) = -\frac{\nu k_1 k_2}{(1-\nu)k_1^2 + k_2^2}\hat{u}_1(k).$$

Substituting this equality in the first component in (2.7) yields

$$\begin{aligned}\hat{\sigma}_{13}(k) &= -\left[\left(\frac{k_2^2}{|k|}+\frac{1}{1-\nu}\frac{k_1^2}{|k|}\right)\hat{u}_1(k)+\frac{\nu}{1-\nu}\frac{k_1k_2}{|k|}\hat{u}_2(k)\right]\\ &= -\frac{|k|^3}{(1-\nu)k_1^2+k_2^2}\hat{u}_1(k)=\mathcal{F}(W'(u_1)).\end{aligned} \tag{2.14}$$

Therefore, the 2D system (2.8) is reduced to a new 1D nonlocal equation

$$\begin{aligned}\mathcal{L}u_1(x_1,x_2) &= -W'(u_1(x_1,x_2)), \quad (x_1,x_2)\in\Gamma,\\ \lim_{x_1\to\pm\infty} u_1(x_1,x_2) &= \pm 1, \quad x_2\in\mathbb{R},\end{aligned} \tag{2.15}$$

where the nonlocal operator $\mathcal{L}$ has the Fourier symbol

$$\frac{|k|^3}{(1-\nu)k_1^2+k_2^2}\in\Big[\frac{|k|}{2}\ ,\ 2|k|\Big], \qquad \nu\in[-1,\frac{1}{2}].$$

Later in Sect. 3, we will derive the integral formulation of the nonlocal operator $\mathcal{L}$ and study the properties of its kernel. Compared to (2.10), we also have the corresponding (further) reduced energy E_Γ^0 on Γ

$$E_\Gamma^0 := \frac{1}{2}\int_{\mathbb{R}^2} u_1\mathcal{L}u_1\,\mathrm{d}x+\int_{\mathbb{R}^2} W(u_1)\,\mathrm{d}x, \tag{2.16}$$

which is equivalent to E_Γ in (2.10) in the perturbed or local sense; see detailed arguments for the perturbed sense in [12]. Notice that the nonlinearity is now coupled to only u_1 on Γ. The main goal is to study the existence, uniqueness, and the properties of solutions to (2.15). If one can solve (2.15), then by the elastic extension introduced in [12], we obtain the vector-field solutions to the original 3D full system (2.6).

Recall that the straight solution (uniform in x_2), i.e., $u_1(x_1,x_2)=\phi(x_1)$, $u_2(x_1,x_2)=0$ is also a solution to (2.15), where $\phi(x_1)$ is the solution to the 1D problem (2.11). For notation simplicity, from now on, we replace $u_1(x_1,x_2)$ with a scalar function $u(x):\mathbb{R}^2\to\mathbb{R}$ in (2.15) and recast (2.15) to

$$\begin{aligned}\mathcal{L}u(x) &= -W'(u(x)), \quad x=(x_1,x_2)\in\mathbb{R}^2,\\ \lim_{x_1\to\pm\infty} u(x_1,x_2) &= \pm 1, \quad x_2\in\mathbb{R}.\end{aligned} \tag{2.17}$$

We will focus on the kernel representation of the operator $\mathcal{L}$ in Sect. 3 and then prove that the solution $u(x)$ to (2.17) must have a 1D profile in Sect. 4. As a consequence, we will finally prove that the straight dislocation is the only stable solution (up to a rotation and translations) to the full system (2.6) in Theorem 4.6,

3 Positive and anisotropic kernel of $\mathcal{L}$

In this section, we derive the integral formulation of the operator $\mathcal{L}$ in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ and prove certain properties of its singular kernel. We will use this integral formulation for $\mathcal{L}$ whenever the singular integration makes sense, for instance, on the space $\{u\in\dot{H}^s(\mathbb{R}^2)$ for any $s\geq 1\}$. In the remaining part of this paper, C is a generic constant whose value may change from line to line.

Recall the integral formulation of the half Laplacian $\Lambda := (-\Delta)^{\frac{1}{2}}$ on $\mathcal{S}(\mathbb{R}^2)$

$$\Lambda u = -\frac{C_d}{2} \int_{\mathbb{R}^2} \big(u(x+y) + u(x-y) - 2u(x)\big)|y|^{-3}\,\mathrm{d}y,$$

where

$$C_d := \frac{2}{\pi} \frac{\Gamma(\frac{3}{2})}{|\Gamma(-\frac{1}{2})|} = \frac{1}{2\pi}.$$

First we state a lemma for the solution to an elliptic equation, whose proof will be given later.

Lemma 3.1 *Let $\beta := 1 - \nu \in [\frac{1}{2}, 2]$. The elliptic equation*

$$\Delta P(x_1, x_2) = \frac{1}{(\beta x_1^2 + x_2^2)^{\frac{5}{2}}}, \quad (x_1, x_2) \in \mathbb{R}^2 \backslash \{0\} \tag{3.1}$$

has a solution

$$P(x_1, x_2) = \frac{v(\theta)}{(x_1^2 + x_2^2)^{\frac{3}{2}}},$$

where $\theta = \arctan \frac{x_2}{x_1}$ and $v(\theta)$ is the unique π-periodic solution to

$$v'' + 9v = (\beta \cos^2\theta + \sin^2\theta)^{-\frac{5}{2}}. \tag{3.2}$$

Moreover, we have the following properties of $v(\theta)$:

(i) $v(\theta)$ is symmetric with respect to $\frac{\pi}{2}$;
(ii) For $\beta \geq 1$, $v(\theta)$ is increasing in $[0, \frac{\pi}{2}]$ and decreasing in $[\frac{\pi}{2}, \pi]$; while for $0 < \beta < 1$, $v(\theta)$ is decreasing in $[0, \frac{\pi}{2}]$ and increasing in $[\frac{\pi}{2}, \pi]$;
(iii) For $\frac{2}{3} < \beta < \frac{3}{2}$, $v(\theta)$ is positive and $\frac{1}{9} c_\beta \leq v(\theta) \leq \frac{1}{9}$ for $0 \leq \theta \leq \pi$, where

$$c_\beta := \min\{\frac{3\beta - 2}{\beta^2}, \frac{3 - 2\beta}{\beta^{\frac{3}{2}}}\} > 0.$$

In Proposition 3.2, we derive the corresponding integral formulation for $\mathcal{L}$ and then study the properties of the singular kernel in Proposition 3.3.

Proposition 3.2 *The integral formulation of $\mathcal{L}$ is given by*

$$\mathcal{L}u = -\frac{1}{4\pi} \int_{\mathbb{R}^2} \big(u(x+y) + u(x-y) - 2u(x)\big) K(y)\,\mathrm{d}y,$$

where $K(y) := 9P(y_1/\sqrt{\beta}, y_2)$ satisfies

$$(\beta\partial_1^2 + \partial_2^2)K(y) = \frac{9}{|y|^5}, \quad \forall y \in \mathbb{R}^2 \backslash \{0\}. \tag{3.3}$$

Proof Step 1. We first derive the integral formulation of Λ^3, where $\Lambda = (-\Delta)^{1/2}$. For any u in the Schwartz class $\mathcal{S}(\mathbb{R}^2)$, we have

$$\begin{aligned}\Lambda^3 u(x) &= \frac{1}{4\pi}\int_{\mathbb{R}^2}\big(\Delta_x u(x+y)+\Delta_x u(x-y)-2\Delta_x u(x)\big)|y|^{-3}\,\mathrm{d}y\\ &= \frac{1}{4\pi}\int_{\mathbb{R}^2}\Delta_y\Big(u(x+y)+u(x-y)-2u(x)-\sum_{i=1,2}y_i^2(\partial_i^2 u)(x)\Big)|y|^{-3}\,\mathrm{d}y\\ &= \lim_{\varepsilon\to 0}\frac{1}{4\pi}\int_{B_\varepsilon^c}\Delta_y\Big(u(x+y)+u(x-y)-2u(x)-\sum_{i=1,2}y_i^2(\partial_i^2 u)(x)\Big)|y|^{-3}\,\mathrm{d}y\\ &= \lim_{\varepsilon\to 0}\left[\frac{9}{4\pi}\int_{B_\varepsilon^c}\Big(u(x+y)+u(x-y)-2u(x)-\sum_{i=1,2}y_i^2(\partial_i^2 u)(x)\Big)|y|^{-5}\,\mathrm{d}y+I_1\right],\end{aligned}$$

where we applied Green's identity in the last equality and I_1 is the boundary term. Since

$$|u(x+y)+u(x-y)-2u(x)-\sum_{i=1,2}y_i^2(\partial_i^2 u)(x)|\le c|y|^4,$$

we have $I_1\sim O(\varepsilon)\to 0$ as $\varepsilon\to 0$. Therefore, we obtain

$$\Lambda^3 u(x) = \frac{9}{4\pi}\int_{\mathbb{R}^2}\big(u(x+y)+u(x-y)-2u(x)-\sum_{i=1,2}y_i^2(\partial_i^2 u)(x)\big)|y|^{-5}\,\mathrm{d}y.$$

Step 2. We show that for any $u\in\mathcal{S}$,

$$\mathcal{F}^{-1}\Big(\frac{|k|^3}{\beta k_1^2+k_2^2}\hat{u}\Big) = -\frac{1}{4\pi}\int_{\mathbb{R}^2}\big(u(x+y)+u(x-y)-2u(x)\big)K(y)\,dy,$$

where $\beta = 1-\nu$ and $K(y)$ satisfies (3.3).

Recall that $P(y)$ is the solution to (3.1) obtained in Lemma 3.1, and thus $K(y)\sim|y|^{-3}$ is homogeneous of degree -3. By using a cutoff near the origin and the dominated convergence theorem, we may assume that $\hat{u}$ vanishes near the origin. Let $u = Lv$, where v is also in $\mathcal{S}$ and the second-order operator $L := -\beta\partial_1^2-\partial_2^2$ has the symbol $\beta k_1^2+k_2^2$. In other words,

$$\hat{v} = \frac{\hat{u}}{\beta k_1^2+k_2^2}.$$

It suffices to show that

$$\begin{aligned}\mathcal{F}^{-1}(|k|^3\hat{v}) = \Lambda^3 v(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^2}\big(&L_x v(x+y)+L_x v(x-y)\\ &-2L_x v(x)\big)K(y)\,\mathrm{d}y.\end{aligned}\tag{3.4}$$

By using a similar computation, the right-hand side above is equal to

$$\begin{aligned}&-\frac{1}{4\pi}\int_{\mathbb{R}^2}L_y\big(v(x+y)+v(x-y)-2v(x)-\sum_{i=1,2}y_i^2(\partial_i^2 v)(x)\big)K(y)\,\mathrm{d}y\\ &= \lim_{\varepsilon\to 0}-\frac{1}{4\pi}\int_{B_\varepsilon^c}L_y\big(v(x+y)+v(x-y)-2v(x)-\sum_{i=1,2}y_i^2(\partial_i^2 v)(x)\big)K(y)\,\mathrm{d}y \qquad (3.5)\\ &= \lim_{\varepsilon\to 0}-\frac{1}{4\pi}\left[\int_{B_\varepsilon^c}\big(v(x+y)+v(x-y)-2v(x)-\sum_{i=1,2}y_i^2(\partial_i^2 v)(x)\big)L_y K(y)\,dy+I_2\right],\end{aligned}$$

where we applied Green's identity in the last equality and I_2 is the boundary term. As before, $I_2 \sim O(\varepsilon) \to 0$ as $\varepsilon \to 0$. Because $L_y K(y) = -9|y|^{-5}$ for $y \neq 0$, the last limit in (3.5) is equal to

$$\frac{9}{4\pi}\int_{\mathbb{R}^2}\Big(v(x+y)+v(x-y)-2v(x)-\sum_{i=1,2} y_i^2(\partial_i^2 v)(x)\Big)|y|^{-5}\,\mathrm{d}y = \Lambda^3 v(x),$$

which yields (3.4). □

Combining Lemma 3.1 and Proposition 3.2, we obtain an anisotropic kernel K. Since $P(y) = \frac{1}{9}K(\sqrt{\beta}y_1, y_2)$ solves (3.1), by a change of variables

$$(\bar{x}_1, \bar{x}_2) = (\frac{1}{\sqrt{\beta}}x_1, x_2), \quad \bar{u}(\bar{x}_1, \bar{x}_2) := u(\sqrt{\beta}\bar{x}_1, \bar{x}_2),$$

we know that if $u(x_1, x_2)$ is a solution to (2.17), then $\bar{u}(\bar{x}_1, \bar{x}_2)$ is a solution to

$$\begin{aligned} \bar{\mathcal{L}}\bar{u} &= -\frac{1}{\sqrt{\beta}}W'(\bar{u}), \quad \bar{x} \in \mathbb{R}^2, \\ \lim_{\bar{x}_1 \to \pm\infty} \bar{u}(\bar{x}_1, \bar{x}_2) &= \pm 1, \quad \bar{x}_2 \in \mathbb{R}, \end{aligned} \tag{3.6}$$

where the nonlocal operator $\bar{\mathcal{L}}$ is given by

$$\bar{\mathcal{L}}\bar{u} = -\frac{1}{4\pi}\int_{\mathbb{R}^2}\big(\bar{u}(\bar{x}+\bar{y})+\bar{u}(\bar{x}-\bar{y})-2\bar{u}(\bar{x})\big)\bar{K}(\bar{y})\,\mathrm{d}\bar{y}, \quad \bar{K}(\bar{y}) := \frac{9v(\theta)}{|\bar{y}|^3}, \tag{3.7}$$

with $v(\theta) = v(\arctan\frac{\bar{y}_2}{\bar{y}_1})$ being the solution to (3.2). In the next section, we will focus on the analysis of the solution to (3.6) and drop the bar in (3.6). Now we summarize the properties of the kernel $\bar{K}$ below.

Proposition 3.3 *For $\frac{2}{3} < \beta < \frac{3}{2}$, the kernel $\bar{K}$ of $\bar{\mathcal{L}}$ in (3.7) is positive and satisfies the following properties:*

(i) $\bar{K}(-x) = \bar{K}(x)$, $\bar{K}(ax) = a^{-3}\bar{K}(x)$ *for any* $a > 0$;
(ii) $0 < \frac{c_\beta}{|x|^3} \leq \bar{K}(x) \leq \frac{1}{|x|^3}$;
(iii) $\max\{|x||\partial_e\bar{K}|, |x|^2|\partial_{ee}\bar{K}|\} \leq \frac{C}{|x|^3}$

for any $x \in \mathbb{R}^2\backslash\{0\}$ and unit vector $e \in S^1$, where c_β is defined in Lemma 3.1(iii).

Corollary 3.4 (Strict positivity property at global minima and global maxima) *For any function $g(\mathbf{w}) \in C(\mathbb{R}^2)$, let $\mathbf{w}_m = (x_m, y_m)$, $\mathbf{w}_M = (x_M, y_M) \in \mathbb{R}^2$ be the points at which $g(\mathbf{w})$ attains it global minimum and maximum respectively. Then we have*

$$\bar{\mathcal{L}}g(\mathbf{w})|_{\mathbf{w}=\mathbf{w}_m} < 0, \quad \bar{\mathcal{L}}g(\mathbf{w})|_{\mathbf{w}=\mathbf{w}_M} > 0$$

provided $g(\mathbf{w})$ is not a constant.

Proof From the positivity of the kernel $\bar{K}$ in Proposition 3.3, since $g(\mathbf{w}_m) \leq g(\mathbf{w})$ for all $\mathbf{w} \in \mathbb{R} \times \mathbb{T}$, we have

$$\bar{\mathcal{L}}g(\mathbf{w})|_{\mathbf{w}=\mathbf{w}_m} \leq 0$$

and the equality holds if and only if $g(\mathbf{w}) \equiv g(\mathbf{w}_m)$ for all $\mathbf{w} \in \mathbb{R} \times \mathbb{T}$. The proof for $\bar{\mathcal{L}}g$ at $\mathbf{w}_M$ is the same. □

We finish this section by proving Lemma 3.1.

Proof of Lemma 3.1 Step 1. To solve

$$\Delta P(x_1, x_2) = \frac{1}{(\beta x_1^2 + x_2^2)^{\frac{5}{2}}}, \quad (x_1, x_2) \in \mathbb{R}^2 \backslash \{0\}, \tag{3.8}$$

by a change of variables

$$x_1 = r\cos\theta, \;\; x_2 = r\sin\theta, \quad P(x, y) = r^{-3} v(\theta) \tag{3.9}$$

in (3.8), we have the ODE (3.2) for $v(\theta)$, i.e.,

$$v'' + 9v = (\beta\cos^2\theta + \sin^2\theta)^{-\frac{5}{2}},$$

where $\beta \in [\frac{1}{2}, 2]$. Notice that the natural period for the harmonic oscillation $v'' + 9v = 0$ is $\frac{2\pi}{3}$ while the force term

$$f(\theta) := (\beta\cos^2\theta + \sin^2\theta)^{-\frac{5}{2}} = \left[\frac{\beta+1}{2} + \frac{\beta-1}{2}\cos(2\theta)\right]^{-\frac{5}{2}}$$

has period π. Therefore, we always have a 2π-periodic solution to the ODE (3.2). Besides, from elementary calculations, one can check that for $\beta \le 1$,

$$f_{\min} = f\left(\frac{\pi}{2} + k\pi\right), \quad f_{\max} = f(k\pi), \quad k \in \mathbb{Z},$$

while for $\beta \ge 1$,

$$f_{\max} = f\left(\frac{\pi}{2} + k\pi\right), \quad f_{\min} = f(k\pi), \quad k \in \mathbb{Z}.$$

Step 2. Existence and uniqueness of a π-periodic solution.

First, we know that P satisfies $P(-x, -y) = P(x, y)$, which, together with (3.9), yields the periodicity $v(\theta + \pi) = v(\theta)$. Therefore, we seek a periodic solution to (3.2) with period π.

Second, by the method of variation of parameters, one can find a special solution $v_0(\theta)$

$$\begin{aligned} v_0(\theta) =& u_1(\theta)\cos(3\theta) + u_2(\theta)\sin(3\theta) \\ & \text{with } u_1(\theta) = -\frac{1}{3}\int_0^\theta \sin(3x) f(x)\,\mathrm{d}x, \; u_2(\theta) = \frac{1}{3}\int_0^\theta \cos(3x) f(x)\,\mathrm{d}x \end{aligned} \tag{3.10}$$

and thus the general solution to (3.2) is given by

$$v(\theta) = C_1\cos(3\theta) + C_2\sin(3\theta) + v_0(\theta). \tag{3.11}$$

Notice that for any π-periodic function $v(\theta)$, we have

$$\int_{-\pi}^{\pi} v(\theta)\cos(k\theta)\,\mathrm{d}\theta = 0, \quad \int_{-\pi}^{\pi} v(\theta)\sin(k\theta)\,\mathrm{d}\theta = 0 \quad \text{for any odd integer } k. \tag{3.12}$$

Therefore, to obtain a π-periodic solution, we must set

$$C_1 := -\frac{1}{\pi}\int_{-\pi}^{\pi} v_0(\theta)\cos(3\theta)\,\mathrm{d}\theta, \quad C_2 := -\frac{1}{\pi}\int_{-\pi}^{\pi} v_0(\theta)\sin(3\theta)\,\mathrm{d}\theta. \tag{3.13}$$

Third, we check $v(0) = v(\pi)$ and $v'(0) = v'(\pi)$.

By plugging in, we have

$$v(0) = v_0(0) + C_1 = -\frac{1}{\pi}\int_{-\pi}^{\pi} v_0(\theta)\cos(3\theta)\,\mathrm{d}\theta$$

and

$$v(\pi) = v_0(\pi) - C_1 = \frac{1}{3}\int_0^{\pi} \sin(3x) f(x)\,\mathrm{d}x + \frac{1}{\pi}\int_{-\pi}^{\pi} v_0(\theta)\cos(3\theta)\,\mathrm{d}\theta.$$

From (3.10) we know that $u_1(\theta)$, $u_2(\theta)$, and thus $v_0(\theta)$ are all periodic functions with period 2π. Hence by integration by parts, we have

$$\begin{aligned}\int_{-\pi}^{\pi} v_0(\theta)\cos(3\theta)\,\mathrm{d}\theta &= -\frac{1}{3}\int_{-\pi}^{\pi} v_0'(\theta)\sin(3\theta)\,\mathrm{d}\theta\\ &= -\frac{1}{3}\int_{-\pi}^{\pi} [u_1(\theta)(-3\sin(3\theta)) + u_2(\theta)(3\cos 3\theta)]\sin(3\theta)\,\mathrm{d}\theta\\ &= \int_{-\pi}^{\pi} u_1(\theta)\sin^2(3\theta) - u_2(\theta)\cos(3\theta)\sin(3\theta)\,\mathrm{d}\theta.\end{aligned}$$

Since $f(x)$ has period π and using (3.12), one can check

$$\begin{aligned}&\int_{-\pi}^{\pi} u_1(\theta)\sin^2(3\theta)\,\mathrm{d}\theta = -\frac{1}{3}(\frac{\theta}{2} - \frac{\sin(6\theta)}{12})\int_0^{\theta}(\sin(3x))f(x)\,\mathrm{d}x\Big|_{-\pi}^{\pi} + \frac{1}{6}\int_{-\pi}^{\pi}\theta\sin(3\theta)f(\theta)\,\mathrm{d}\theta\\ &= -\frac{\pi}{6}\int_0^{\pi}(\sin(3\theta))f(\theta)\,\mathrm{d}\theta,\end{aligned} \tag{3.14}$$

where we used

$$\int_{-\pi}^{\pi}\theta\sin(3\theta)f(\theta)\,\mathrm{d}\theta = \pi\int_0^{\pi}\sin(3\theta)f(\theta)\,\mathrm{d}\theta.$$

Similarly, we obtain

$$\int_{\pi}^{\pi} u_2(\theta)\cos(3\theta)\sin(3\theta)\,\mathrm{d}\theta = 0.$$

Therefore, we conclude that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} v_0(\theta)\cos(3\theta)\,\mathrm{d}\theta = -\frac{1}{6}\int_0^{\pi}\sin(3\theta)f(\theta)\,\mathrm{d}\theta, \tag{3.15}$$

which yields

$$v(0) = \frac{1}{6}\int_0^{\pi}\sin(3x)f(x)\,\mathrm{d}x = v(\pi).$$

Then by plugging in, we have

$$v'(0) = v_0'(0) + 3C_2 = -\frac{3}{\pi}\int_{-\pi}^{\pi} v_0(\theta)\sin(3\theta)\,\mathrm{d}\theta$$

and

$$v'(\pi) = v_0'(\pi) - 3C_2 = -\int_0^{\pi}\cos(3\theta)f(\theta)\,\mathrm{d}\theta + \frac{3}{\pi}\int_{-\pi}^{\pi} v_0(\theta)\sin(3\theta)\,\mathrm{d}\theta.$$

By the similar calculation in (3.14), we have

$$\int_{-\pi}^{\pi} v_0(\theta)\sin(3\theta)\,\mathrm{d}\theta = \int_{-\pi}^{\pi} u_2(\theta)\cos^2(3\theta)\,\mathrm{d}\theta = \frac{\pi}{6}\int_0^{\pi}\cos(3\theta)f(\theta)\,\mathrm{d}\theta. \tag{3.16}$$

Therefore, we verified $v'(0) = v'(\pi)$. Thus from the uniqueness of the solution to ODE (3.2) we conclude that (3.11) with coefficients in (3.13) is the unique π-periodic solution to (3.2). From (3.15) and (3.16), we have

$$\begin{aligned} v(\theta) =& \frac{1}{6}\left(2\int_0^{\theta}\cos(3x)f(x)\,\mathrm{d}x - \int_0^{\pi}\cos(3x)f(x)\,\mathrm{d}x\right)\sin(3\theta) \\ &+ \frac{1}{6}\left(-2\int_0^{\theta}\sin(3x)f(x)\,\mathrm{d}x + \int_0^{\pi}\sin(3x)f(x)\,\mathrm{d}x\right)\cos(3\theta) \\ =& \frac{1}{6}\left(\int_0^{\theta}\cos(3x)f(x)\,\mathrm{d}x - \int_{\theta}^{\pi}\cos(3x)f(x)\,\mathrm{d}x\right)\sin(3\theta) \\ &- \frac{1}{6}\left(\int_0^{\theta}\sin(3x)f(x)\,\mathrm{d}x - \int_{\theta}^{\pi}\sin(3x)f(x)\,\mathrm{d}x\right)\cos(3\theta) \end{aligned}$$

and

$$\begin{aligned} v'(\theta) =& \frac{\cos(3\theta)}{2}\left(\int_0^{\theta}\cos(3x)f(x)\,\mathrm{d}x - \int_{\theta}^{\pi}\cos(3x)f(x)\,\mathrm{d}x\right) \\ &+ \frac{\sin(3\theta)}{2}\left(\int_0^{\theta}\sin(3x)f(x)\,\mathrm{d}x - \int_{\theta}^{\pi}\sin(3x)f(x)\,\mathrm{d}x\right) \end{aligned}$$

for $0 \le \theta \le \pi$.

Step 3. Properties of $v(\theta)$ and the range of β such that v is positive.

Denote $g_1(x) := \cos(3x)f(x)$ and $g_2(x) := \sin(3x)f(x)$, which have the symmetric property

$$g_1(\frac{\pi}{2}+x) = -g_1(\frac{\pi}{2}-x), \quad g_2(\frac{\pi}{2}+x) = g_2(\frac{\pi}{2}-x).$$

Therefore, for $0 \le \theta < \pi$, we have

$$\left(\int_0^{\theta} - \int_{\theta}^{\pi}\right) g_1(x)\,\mathrm{d}x = 2\int_0^{\theta} g_1(x)\,\mathrm{d}x, \quad \left(\int_0^{\theta} - \int_{\theta}^{\pi}\right) g_2(x)\,\mathrm{d}x = -2\int_{\theta}^{\frac{\pi}{2}} g_2(x)\,\mathrm{d}x$$

and thus $v(\theta)$ and $v'(\theta)$ can be expressed as

$$\begin{aligned} v(\theta) &= \frac{\sin(3\theta)}{3}\int_0^{\theta} g_1(x)\,\mathrm{d}x + \frac{\cos(3\theta)}{3}\int_{\theta}^{\frac{\pi}{2}} g_2(x)\,\mathrm{d}x, \quad 0 \le \theta < \pi, \\ v'(\theta) &= \cos(3\theta)\int_0^{\theta} g_1(x)\,\mathrm{d}x - \sin(3\theta)\int_{\theta}^{\frac{\pi}{2}} g_2(x)\,\mathrm{d}x, \quad 0 \le \theta < \pi. \end{aligned}$$

Moreover, we have

$$v(\theta) = v(\pi - \theta), \quad v'(\theta) = -v'(\pi - \theta). \tag{3.17}$$

Now we give the following claim:

For $0 \le \theta < \pi$, the equation $v'(\theta) = 0$ only has two roots $\theta = 0, \pi$.

Proof Indeed, we only need to prove this claim for $\beta > 1$. For the case $0 < \beta < 1$, denote $\bar{\beta} := \frac{1}{\beta}$, then by $v'(\frac{\pi}{2} - \theta, \beta) = \bar{\beta}^{\frac{5}{2}} v'(\theta, \bar{\beta})$, the problem is reduced to the case $\beta > 1$.

Denote $w(\theta) := v'(\theta)$. Then w satisfies $w'' + 9w = f'(\theta)$. For $\beta > 1$, we know that $f'(\theta) > 0$ in $(0, \pi/2)$. By the symmetric property for $v'(\theta)$ in (3.17), it remains to prove that the solution to

$$w'' + 9w = f'(\theta) > 0, \quad w(0) = w(\frac{\pi}{2}) \tag{3.18}$$

is strictly positive for $0 < \theta < \frac{\pi}{2}$. If it is not true, then there exist $0 < a \leq b < \frac{\pi}{2}$ such that $w(a) = w(b) = 0$ and $w(\theta) > 0$ for $\theta \in (0, a) \cup (b, \frac{\pi}{2})$. Notice that the eigenvalue problem

$$w'' + \lambda w = 0, \quad w(0) = w(a) = 0$$

has the smallest eigenvalue $\lambda_1 = \left(\frac{\pi}{a}\right)^2$. If $a \leq \frac{\pi}{3}$, then $\lambda_1 \geq 9$. However, this is impossible because $w(\theta) > 0$ for $\theta \in (0, a)$ and (3.18) implies

$$\int_0^a w(w'' + \lambda_1 w)\, d\theta \geq \int_0^a w(w'' + 9w)\, d\theta = \int_0^a f'(\theta) w\, d\theta > 0.$$

Thus we conclude that $a > \frac{\pi}{3}$. Similarly, since the eigenvalue problem

$$w'' + \lambda w = 0, \quad w(b) = w(\frac{\pi}{2}) = 0$$

has the smallest eigenvalue $\lambda_1 = \left(\frac{\pi}{\frac{\pi}{2} - b}\right)^2$, we conclude that $b < \frac{\pi}{6}$. This gives a contradiction and we complete the proof. □

Then by elementary calculations, we have

$$\begin{aligned} v(0) =& \frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{\sin(3x)}{(\beta \cos^2 x + \sin^2 x)^{\frac{5}{2}}}\, dx = -\frac{1}{3} \int_0^1 \frac{1 - 4t^2}{(1 + (\beta - 1)t^2)^{\frac{5}{2}}}\, dt \\ \overset{t = \frac{\tan y}{\sqrt{\beta - 1}}}{=} & -\frac{1}{3} \int_0^{\arctan \sqrt{\beta - 1}} \frac{1}{\sqrt{\beta - 1}} \left(1 - \frac{4}{\beta - 1} \tan^2 y\right) \frac{1}{\sec^3 y}\, dy \\ \overset{s = \sin y}{=} & -\frac{1}{3} \int_0^{\frac{\sqrt{\beta - 1}}{\sqrt{\beta}}} \frac{1}{\sqrt{\beta - 1}} \left(1 - \frac{\beta + 3}{\beta - 1} s^2\right) ds = \frac{3 - 2\beta}{9\beta^{\frac{3}{2}}}. \end{aligned}$$

Similarly, we have

$$v\left(\frac{\pi}{2}\right) = -\frac{1}{3} \int_0^{\frac{\pi}{2}} \frac{\cos(3x)}{(\beta \cos^2 x + \sin^2 x)^{\frac{5}{2}}}\, dx = -\frac{1}{3} \int_0^1 \frac{1 - 4t^2}{(\beta + (1 - \beta)t^2)^{\frac{5}{2}}}\, dt = \frac{3\beta - 2}{9\beta^2}.$$

On one hand, if $\frac{1}{2} \leq \beta \leq 1$,

$$\frac{3\beta - 2}{9\beta^2} = v\left(\frac{\pi}{2}\right) \leq v(\theta) \leq v(0) = \frac{3 - 2\beta}{9\beta^{\frac{3}{2}}} \quad \text{for any } 0 \leq \theta \leq \pi.$$

In this case, $v(\frac{\pi}{2}) = 0$ if and only if $\beta = \frac{2}{3}$ and thus

$$v_{\min} = \frac{3\beta - 2}{9\beta^2} > 0, \quad v_{\max} = \frac{3 - 2\beta}{9\beta^{\frac{3}{2}}} \leq \frac{1}{9} \quad \text{for } \frac{2}{3} < \beta \leq 1.$$

On the other hand, if $1 \le \beta \le 2$,

$$\frac{3\beta-2}{9\beta^2} = v(\frac{\pi}{2}) \ge v(\theta) \ge v(0) = \frac{3-2\beta}{9\beta^{\frac{3}{2}}} \quad \text{for any } 0 \le \theta \le \pi.$$

In this case, $v(0) = 0$ if and only if $\beta = \frac{3}{2}$ and thus

$$v_{\min} = \frac{3-2\beta}{9\beta^{\frac{3}{2}}} > 0, \quad v_{\max} = \frac{3\beta-2}{9\beta^2} \le \frac{1}{9} \quad \text{for } 1 \le \beta < \frac{3}{2}.$$

Therefore, we conclude that when $\frac{2}{3} < \beta < \frac{3}{2}$, there exists a unique π-periodic positive solution $v(\theta)$ to (3.2). □

4 Bounded stable solution has a 1D profile

In this section, we will prove that any bounded stable solution to (3.6), dropping bars for notation simplicity, has a 1D profile, i.e., $u(x) = \phi(e \cdot x)$ for some $e \in S^1$, where ϕ is the unique (up to translations) solution to a 1D problem; see Theorem 4.6. From [5,16], we know that ϕ is bounded, increasing from -1 to 1, and a local minimizer of the corresponding energy. The proof relies on the local BV estimates originally developed in [6] to study the quantitative flatness of nonlocal minimal surface. Their method does not use any extension argument and thus is particularly powerful for the nonlocal problem with general anisotropic kernel. This is the key in our case as we do not have a scalar-valued extended 3D problem. In this section, we will always assume $\frac{2}{3} < \beta < \frac{3}{2}$ so that we have good properties of the kernel $\bar{K}$ in Proposition 3.3.

The proof of the 1D profile is divided into the following three subsections, which roughly say that stability implies flatness; c.f. [6,7,10,13,18]. First, let us clarify the definition of stable solutions and how to define the perturbations to these stable solutions in a ball B_R with respect to some direction $\boldsymbol{v}$. Define the total energy of u in any ball $B_R \subset \mathbb{R}^2$ as

$$\begin{aligned} E^0_\Gamma(u; B_R) :&= \frac{C_d}{4} \iint_{\mathbb{R}^2\times\mathbb{R}^2 \setminus B_R^c \times B_R^c} |u(x)-u(y)|^2 \bar{K}(x-y)\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{\sqrt{\beta}} \int_{B_R} W(u(x))\,\mathrm{d}x \\ &=: \frac{C_d}{4}\mathcal{E}(u; B_R) + F(u; B_R), \end{aligned} \tag{4.1}$$

where $\bar{K}$ is the kernel in (3.7) satisfying the properties in Proposition 3.3. Here the nonlocal energy $\mathcal{E}(u; B_R)$ can be viewed as the contribution in B_R of the semi-norm $\|\cdot\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^2)}$ because we formally have

$$\|u\|^2_{\dot{H}^{\frac{1}{2}}} = \lim_{R\to+\infty} \mathcal{E}(u; B_R).$$

Definition 2 We say that u is a stable solution to (3.6) if the second local variation of E^0_Γ defined in (2.16) is nonnegative, i.e.,

$$\int_{\mathbb{R}^2} \left(\bar{\mathcal{L}}v + \frac{1}{\sqrt{\beta}} W''(u)v \right) v\,\mathrm{d}x \ge 0 \quad \text{for any } v \in C_c^2(\mathbb{R}^2).$$

Next, following [6] we define the perturbations of these stable solutions in a ball B_R with respect to some direction $\boldsymbol{v}$. Let $R \ge 1$, define the perturbed coordinates along $\boldsymbol{v} \in S^1$

direction as

$$\psi_{t,\boldsymbol{v}}(z) := z + t\varphi(z)\boldsymbol{v},$$

where φ is a cut-off function compact supported in B_R

$$\varphi(z) = \begin{cases} 1, & |z| \le \frac{R}{2}, \\ 2 - \frac{2|z|}{R}, & \frac{R}{2} \le |z| \le R, \\ 0, & |z| \ge R. \end{cases}$$

Since for t small enough, $\psi_{t,\boldsymbol{v}}$ is invertible, the local perturbed solution is defined by the pushforward operator

$$P_{t,\boldsymbol{v}}u(x) = u(\psi_{t,\boldsymbol{v}}^{-1}(x)).$$

Based on the local perturbed solutions above, we define the discrete second variation of $E^0_\Gamma(u, B_R)$ as

$$\Delta^t_{\boldsymbol{vv}} E^0_\Gamma(u, B_R) := E^0_\Gamma(P_{t,\boldsymbol{v}}u, B_R) + E^0_\Gamma(P_{-t,\boldsymbol{v}}u, B_R) - 2E^0_\Gamma(u, B_R).$$

4.1 Interior BV estimate

The interior BV estimate follows the spirit of [6], which gives a quantitative flatness estimate in B_1 for a stable set in B_R. Let us first give the estimate of the discrete second variation of the energy $\Delta^t_{\boldsymbol{vv}} E^0_\Gamma(u, B_R)$ in Lemma 4.1 and an identity for the nonlocal energy $\mathcal{E}$ in Lemma 4.2.

Lemma 4.1 *Let $\bar{K}$ be the kernel in (4.1) satisfying the properties in Proposition 3.3. Then the discrete second variation of the energy $\Delta^t_{\boldsymbol{vv}} E^0_\Gamma(u, B_R)$ satisfies the estimate*

$$\Delta^t_{\boldsymbol{vv}} E^0_\Gamma(u, B_R) \le C\frac{t^2}{R^2}\mathcal{E}(u, B_R) \quad \textit{for any } R \ge 1,$$

where C is a constant.

The proof of this lemma is given by [6, Lemma 2.1] (see also [10, Lemma 2.1] and [13, Lemma 3.2]). Recall that $\bar{K}$ satisfies properties (i–iii) in Proposition 3.3.

Next, we recall an identity for nonlocal energy, which is originally introduced in [16] and crucially used in [6] for the interior BV estimate. Note that it does not depend on exact formulas of the kernel $\bar{K}$ as long as the integrals are well-defined. In the remaining context, $f_+(x) := \max\{f(x), 0\}$ and $f_-(x) := -\min\{f(x), 0\}$.

Lemma 4.2 *Let u, v be any measurable functions such that $\mathcal{E}(u, B_R) < \infty$ and $\mathcal{E}(v, B_R) < \infty$. Then we have*

$$\begin{aligned} &\mathcal{E}(u, B_R) + \mathcal{E}(v, B_R) \\ &\quad = \mathcal{E}(\min\{u, v\}, B_R) + \mathcal{E}(\max\{u, v\}, B_R) \\ &\qquad + 2\iint_{\mathbb{R}^2\times\mathbb{R}^2\setminus B_R^c\times B_R^c} (v-u)_+(x)(v-u)_-(y)\bar{K}(x-y)\,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

where $\bar{K}$ is the kernel associated with the nonlocal energy $\mathcal{E}$.

Now we are ready to give the interior BV estimate for stable solutions in Definition 2. The proof is similar to [13, Lemma 3.6] and [10, Lemma 2.2] due to the properties of the kernel $\bar{K}$ in Proposition 3.3. We include the proof for completeness.

Lemma 4.3 *Let $|u| \le M$ be a bounded stable solution to (3.6) satisfying Definition 2. Then there exists a constant $C(\beta, M)$ depending only on β and M such that for any $R \ge 1$,*

$$\left(\int_{B_{\frac{1}{2}}} (\partial_{\boldsymbol{v}} u(x))_+ \,\mathrm{d}x\right)\left(\int_{B_{\frac{1}{2}}} (\partial_{\boldsymbol{v}} u(y))_- \,\mathrm{d}y\right) \le C(\beta, M)\frac{\mathcal{E}(u, B_R)}{R^2}, \tag{4.2}$$

$$\int_{B_{\frac{1}{2}}} |\nabla u(x)| \,\mathrm{d}x \le C(\beta, M)(1 + \sqrt{\mathcal{E}(u, B_1)}). \tag{4.3}$$

Proof Step 1. Proof of (4.2). Denote

$$u_M := \max\{P_{t,\boldsymbol{v}} u, u\} \quad \text{and}$$
$$u_m := \min\{P_{t,\boldsymbol{v}} u, u\}.$$

Then by the identity in Lemma 4.2, we have for $R \ge 1$,

$$\begin{aligned}
&\mathcal{E}(u_m, B_R) + \mathcal{E}(u_M, B_R)\\
&+ 2\int_{B_{\frac{1}{2}}}\int_{B_{\frac{1}{2}}} (u(x - t\boldsymbol{v}) - u(x))_+ (u(y - t\boldsymbol{v}) - u(y))_- \bar{K}(x - y) \,\mathrm{d}x \,\mathrm{d}y\\
&\le \mathcal{E}(u, B_R) + \mathcal{E}(P_{t,\boldsymbol{v}} u, B_R),
\end{aligned} \tag{4.4}$$

where we used $P_{t,\boldsymbol{v}} u(x) = u(x - t\boldsymbol{v})$ for $x \in B_{\frac{1}{2}}$ and $|t|$ small enough. Moreover, for the local term F in total energy, we always have

$$F(u_m, B_R) + F(u_M, B_R) = F(P_{t,\boldsymbol{v}}(u), B_R) + F(u, B_R). \tag{4.5}$$

Since $|x - y| < 1$ for $x, y \in B_{\frac{1}{2}}$ and

$$0 < \frac{c_\beta}{|x - y|^3} \le \bar{K}(x - y)$$

from Proposition 3.3, (4.4) and (4.5) yield

$$\begin{aligned}
&E^0_\Gamma(u_m, B_R) + E^0_\Gamma(u_M, B_R)\\
&+ C(\beta)\int_{B_{\frac{1}{2}}}\int_{B_{\frac{1}{2}}} (u(x - t\boldsymbol{v}) - u(x))_+ (u(y - t\boldsymbol{v}) - u(y))_- \,\mathrm{d}x \,\mathrm{d}y\\
&\le E^0_\Gamma(u, B_R) + E^0_\Gamma(P_{t,\boldsymbol{v}} u, B_R).
\end{aligned}$$

Then by the stability of u and Lemma 4.1, we have

$$\begin{aligned}
&C(\beta)\int_{B_{\frac{1}{2}}}\int_{B_{\frac{1}{2}}} (u(x - t\boldsymbol{v}) - u(x))_+ (u(y - t\boldsymbol{v}) - u(y))_- \,\mathrm{d}x \,\mathrm{d}y\\
&\le \Delta^t_{\boldsymbol{vv}} E^0_\Gamma(u, B_R) - \big[E^0_\Gamma(u_m, B_R) + E^0_\Gamma(u_M, B_R)\\
&+E^0_\Gamma(P_{-t,\boldsymbol{v}} u, B_R) - 3E^0_\Gamma(u, B_R)\big]\\
&\le \Delta^t_{\boldsymbol{vv}} E^0_\Gamma(u, B_R) + o(t^2) \le C(\beta)\frac{t^2}{R^2}\mathcal{E}(u, B_R).
\end{aligned} \tag{4.6}$$

Here in the second inequality, we used the fact that the second variation of E^0_Γ is nonnegative, which implies

$$\begin{aligned}
&[E^0_\Gamma(u_m, B_R) - E^0_\Gamma(u, B_R)] + [E^0_\Gamma(u_M, B_R) - E^0_\Gamma(u, B_R)]\\
&+[E^0_\Gamma(P_{-t,\boldsymbol{v}} u, B_R) - E^0_\Gamma(u, B_R)] \ge -o(t^2).
\end{aligned}$$

Dividing both sides by t^2 in (4.6) and taking $t \to 0$, we conclude (4.2).

Step 2. Proof of (4.3). Denote

$$A^{\pm} := \int_{B_{\frac{1}{2}}} (\partial_{\boldsymbol{v}} u(x))_{\pm} \, \mathrm{d}x.$$

Then (4.2) gives

$$\min\{A^+, A^-\} \le \frac{C(\beta)}{R} \sqrt{\mathcal{E}(u, B_R)}.$$

Thus we have

$$\int_{B_{\frac{1}{2}}} |\partial_{\boldsymbol{v}} u| \, \mathrm{d}x = A^+ + A^- = |A^+ - A^-| + 2\min\{A^+, A^-\} \le C(\beta, M)(1 + \sqrt{\mathcal{E}(u, B_1)}),$$

where we used

$$|A^+ - A^-| = |\int_{B_{\frac{1}{2}}} \partial_{\boldsymbol{v}} u(x) \, \mathrm{d}x| \le \int_{\partial B_{\frac{1}{2}}} |u \boldsymbol{v} \cdot n_{\partial B_{\frac{1}{2}}}| \le C(M)$$

due to boundedness of u. Here $n_{\partial B_{\frac{1}{2}}}$ is the outer unit normal of $\partial B_{\frac{1}{2}}$. Therefore we obtain (4.3) since $|\nabla u| \le |\partial_1 u| + |\partial_2 u|$. □

4.2 Energy estimates in any balls

In this subsection, we will prove the energy estimate in any balls by combining the interior BV estimate in Lemma 4.3 and a sharp interpolation inequality for the nonlocal energy $\mathcal{E}$ below.

Lemma 4.4 *Let $|u| \le M$ be a bounded function. Assume that u is Lipschitz in B_2 with $L_0 := \max\{2, \|\nabla u\|_{L^\infty(B_2)}\}$. Then there exists a constant $C(M)$ depending only on M such that*

$$\mathcal{E}(u, B_1) \le \iint_{\mathbb{R}^2 \times \mathbb{R}^2 \setminus B_1^c \times B_1^c} \frac{|u(x) - u(y)|^2}{|x - y|^3} \, \mathrm{d}x \, \mathrm{d}y \le C(M) \log L_0 (1 + \int_{B_2} |\nabla u| \, \mathrm{d}x).$$

This lemma is proved in [10, Lemma 3.1] for the kernel $\frac{1}{|x|^3}$, and we conclude this lemma for the kernel $\bar{K}$ since $\bar{K}(x) \le \frac{1}{|x|^3}$ due to Proposition 3.3.

With this sharp interpolation lemma and the interior BV estimate in Lemma 4.3, we are ready to obtain the energy estimates in any balls below.

Proposition 4.5 *Let $|u| \le M$ be a bounded stable solution to (3.6) satisfying Definition 2. Assume that W satisfies (2.12) and $L_* := \max\{2, \|W\|_{C_b^{2,\alpha}(\mathbb{R})}\}$. Then there exists constant $C(\beta, M, L_*)$ depending only on β, M, and L such that for any $B_R \subset \mathbb{R}^2$ and $R \ge 1$,*

$$\int_{B_R} |\nabla u| \, \mathrm{d}x \le C(\beta, M, L_*) R \log(L_* R), \qquad \mathcal{E}(u, B_R) \le C(\beta, M, L_*) R \log^2(L_* R) \tag{4.7}$$

Proof First, by interior regularity estimate for $\mathcal{L}$, cf. [1,9], for $L_1 := \|W\|_{C_b^{2,\alpha}(\mathbb{R})}$, we have

$$\|\nabla u\|_{L^\infty(B_2)} \le C(\beta, M, L_1).$$

Denote $L_2 := \max\{2, CL_1\}$. Then combining (4.3) and Lemma 4.4, we have

$$\begin{aligned}\int_{B_{\frac{1}{2}}} |\nabla u(x)|\,\mathrm{d}x &\leq C(\beta, M)\left(1+\sqrt{C(M)\log L_2\left(1+\int_{B_2}|\nabla u|\,\mathrm{d}x\right)}\right)\\ &\leq \frac{C(\beta, M)\log L_2}{\delta}+\delta\int_{B_2}|\nabla u|\,\mathrm{d}x,\end{aligned} \tag{4.8}$$

where we used Young's inequality in the last inequality.

Second, we prove a uniform bound by a scaling argument and a standard iteration argument. For any z, choose $\rho < 1$ such that $B_\rho(z) \subset B_1$ and $\tilde{u}(x) := u(z + \frac{\rho}{2}x)$. Notice that

$$\bar{K}(\frac{2x}{\rho}) = (\rho/2)^3\bar{K}(x)$$

due to Proposition 3.3. Then $\tilde{u}$ satisfies (3.6) with W replaced by $\frac{\rho}{2}W$. Therefore, (4.8) still holds, i.e.,

$$\int_{B_{\frac{1}{2}}} |\nabla\tilde{u}(x)|\,\mathrm{d}x \leq \frac{C(\beta, M)\log 2L_2}{\delta}+\delta\int_{B_2}|\nabla\tilde{u}|\,\mathrm{d}x,$$

which is equivalent to

$$\frac{1}{\rho}\int_{B_{\frac{\rho}{4}}(z)}|\nabla u|\,\mathrm{d}x \leq \frac{C(\beta, M)\log L_2}{\delta}+\frac{\delta}{\rho}\int_{B_\rho(z)}|\nabla u|\,\mathrm{d}x.$$

Then by a standard iteration argument, one obtain

$$\int_{B_{\frac{1}{2}}}|\nabla u|\,\mathrm{d}x \leq C(\beta, M)\log L_2.$$

By the same scaling argument with $u_R(x) := u(z + 2Rx)$, one can obtain

$$\int_{B_R(z)}|\nabla u|\,\mathrm{d}x \leq C(\beta, M)R\log(CRL_2) \quad \text{for any } R \geq 1.$$

Moreover by Lemma 4.4 and scaling argument, we also have

$$\mathcal{E}(u, B_R) \leq C(\beta, M)R\log^2(CRL_2) \quad \text{for any } R \geq 1.$$

Therefore, we conclude (4.7). □

4.3 1D profile conclusion

In this subsection, we are in the position to state and prove that any bounded stable solution to (3.6) has a 1D monotone profile.

Now we give the main theorem in this section, which corresponds to the flatness result for 2D minimal surface with fractional anisotropic perimeters.

Theorem 4.6 *Let $\beta = 1 - \nu \in (\frac{2}{3}, \frac{3}{2})$. Assume that $|u| \leq M$ is a bounded stable solution to (3.6) and W satisfies (2.12). Then u has a 1D monotone profile and $|u| \leq 1$. As a consequence, any bounded stable solution to (2.17) also has a 1D monotone profile and $|u| \leq 1$. Moreover,* *the solution to (2.17) can be characterized as* $u(x) = \phi(e \cdot x)$ *for any* $e := (\cos\alpha, \sin\alpha) \in S^1$ *with* $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$, *where* ϕ *is the unique (up to translations) solution to 1D problem*

$$\begin{aligned} &(-\Delta)^{\frac{1}{2}}\phi(x_1) = -(\beta\cos^2\alpha + \sin^2\alpha)W'(\phi(x_1)), \quad x_1 \in \mathbb{R} \\ &\lim_{x_1 \to \pm\infty} \phi(x_1) = \pm 1. \end{aligned} \tag{4.9}$$

Proof Combining the uniform energy estimate (4.7) with the interior BV estimate (4.2), taking $R \to +\infty$, we know that

$$\left(\int_{B_{\frac{1}{2}}} (\partial_{\boldsymbol{v}} u(x))_+ \,\mathrm{d}x\right)\left(\int_{B_{\frac{1}{2}}} (\partial_{\boldsymbol{v}} u(y))_- \,\mathrm{d}y\right) = 0.$$

Since this is true for any direction $\boldsymbol{v} \in S^1$ and any half ball in $\mathbb{R}^2$, we have

$$\partial_{\boldsymbol{v}} u \geq 0 \text{ in } \mathbb{R}^2 \quad \text{or} \quad \partial_{\boldsymbol{v}} u \leq 0 \text{ in } \mathbb{R}^2 \qquad \text{for any } \boldsymbol{v} \in S^1,$$

which yields the conclusion that u has a 1D monotone profile.

Next, we prove that u is given by $\phi(e \cdot x)$ and ϕ is the solution to the 1D problem (4.9).

Let the direction e be $e = (\cos\alpha, \sin\alpha)$ for some α. Due to the far field boundary condition (2.3), we consider only the case $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Define the rotation matrix

$$R := \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

Define the new coordinates under the rotation matrix as

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} := R^T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} \bar{k}_1 \\ \bar{k}_2 \end{pmatrix} := R^T \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}.$$

Then we find ϕ such that $u(x) = \phi(e \cdot x)$ satisfies (2.14), i.e.,

$$-\mathcal{F}(W'(u))(k) = -\mathcal{F}(W'(\phi(e \cdot x))) = \frac{|k|^3}{\beta k_1^2 + k_2^2}\mathcal{F}(u)(k) = \frac{|k|^3}{\beta k_1^2 + k_2^2}\mathcal{F}(\phi(e \cdot x))(k).$$

This, together with the property that the Fourier transform commutes with rotations, implies

$$\begin{aligned} &\frac{|k|^3}{\beta k_1^2 + k_2^2}\mathcal{F}(\phi(\bar{x}_1))(\bar{k}_1)\delta(\bar{k}_2) \\ &\quad = -\mathcal{F}(W'(\phi(e \cdot x))) = -\mathcal{F}(W'(\phi))(\bar{k}_1)\delta(\bar{k}_2). \end{aligned}$$

Therefore, $\phi(x_1)$ is the solution to (4.9), or equivalently

$$|k_1|\hat{\phi}(k_1) = -\tilde{\beta}\mathcal{F}(W'(\phi))(k_1)$$

with $\tilde{\beta}$ satisfying

$$\frac{|\bar{k}_1|}{\tilde{\beta}} = \frac{|k|^3}{\beta k_1^2 + k_2^2}, \quad \bar{k}_2 = -k_1\sin\alpha + k_2\cos\alpha = 0.$$

Then elementary calculations yield

$$\tilde{\beta} = \beta\cos^2\alpha + \sin^2\alpha.$$

From [5,16], the solution ϕ to (4.9) is unique (up to translations), bounded, increasing from -1 to 1, and a local minimizer of the isotropic nonlocal energy

$$E_\Gamma^i = \frac{1}{2}\int_{\mathbb{R}} u(-\Delta)^{\frac{1}{2}} u \,\mathrm{d}x + \tilde{\beta}\int_{\mathbb{R}} W(u)\,\mathrm{d}x.$$

Thus the second local variation of E_Γ^i is nonnegative; see also [11] for the positivity of the linearized operator $(-\Delta)^{\frac{1}{2}} + \tilde{\beta} W''(\phi) I$. Therefore, $u(x) = \phi(e \cdot x)$ characterizes the bounded stable solutions to (2.17). □

Remark 2 It is easy to verify that the local minimizer of the energy E_Γ^0 is a bounded stable solution to (3.6). From the proof of [13, Remark 1.4] and Theorem 4.6, one also knows that any bounded stable solution for $\frac{2}{3} < \beta < \frac{3}{2}$ has a 1D monotone profile and thus a local minimizer. That is to say, for (3.6) (also (2.17)) with $\frac{2}{3} < \beta < \frac{3}{2}$, bounded stable solutions and local minimizers are the same objects and both are 1D monotone.

Remark 3 Let $\nu = 1-\beta \in (-\frac{1}{2}, \frac{1}{3})$. From the solution u_1 to (2.15), one can further solve the other two components u_2, u_3 by (2.13) and the elastic extension [12] based on the Dirichlet to Neumann map. Finally, the stable solution to the full system (2.6) is completely solved.

Appendix

A Derivation of Euler–Lagrange equation

Proof of Lemma 2.1 From Definition 1 of local minimizers, we calculate the variation of the energy in terms of a perturbation with compact support in an arbitrary ball B_R. For any $\mathbf{v} \in C^\infty(B_R\backslash\Gamma)$ such that $\mathbf{v}$ has compact support in B_R and satisfies (2.5), we consider the perturbation $\delta\mathbf{v}$, where δ is a small real number. We denote $\varepsilon := \varepsilon(\mathbf{u})$, $\sigma := \sigma(\mathbf{u})$ and $\varepsilon_1 := \varepsilon(\mathbf{v})$, $\sigma_1 := \sigma(\mathbf{v})$. Then we have that

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}(E(\mathbf{u}+\delta\mathbf{v}) - E(\mathbf{u}))\\
&\quad= \int_{B_R\backslash\Gamma}\frac{1}{2}(\sigma_1 : \varepsilon + \sigma : \varepsilon_1)\,\mathrm{d}x + \int_{B_R\cap\Gamma}\partial_1 W(u_1^+, u_2^+)v_1^+ + \partial_2 W(u_1^+, u_2^+)v_2^+\,\mathrm{d}\Gamma\\
&\quad= \int_{B_R\backslash\Gamma}\sigma : \varepsilon_1\,\mathrm{d}x + \int_{B_R\cap\Gamma}\partial_1 W(u_1^+, u_2^+)v_1^+ + \partial_2 W(u_1^+, u_2^+)v_2^+\,\mathrm{d}\Gamma\\
&\quad= \int_{B_R\backslash\Gamma}\sigma : \nabla\mathbf{v}\,\mathrm{d}x + \int_{B_R\cap\Gamma}\partial_1 W(u_1^+, u_2^+)v_1^+ + \partial_2 W(u_1^+, u_2^+)v_2^+\,\mathrm{d}\Gamma\\
&\quad= -\int_{B_R\backslash\Gamma}\partial_j\sigma_{ij}v_i\,\mathrm{d}x + \int_{B_R\cap\Gamma}\sigma_{ij}^+ n_j^+ v_i^+\,\mathrm{d}\Gamma\\
&\qquad+ \int_{B_R\cap\Gamma}\sigma_{ij}^- n_j^- v_i^-\,\mathrm{d}\Gamma + \int_{B_R\cap\Gamma}\partial_1 W(u_1^+, u_2^+)v_1^+ + \partial_2 W(u_1^+, u_2^+)v_2^+\,\mathrm{d}\Gamma \geq 0,
\end{aligned}$$

where we used the property that σ and $\nabla\cdot\sigma$ are locally integrable in $\{x_3 > 0\} \cup \{x_3 < 0\}$ when carrying out the integration by parts, and the outer normal vector of the boundary Γ is $\mathbf{n}^+$ (resp. $\mathbf{n}^-$) for the upper half-plane (resp. lower half-plane). Similarly, taking perturbation as $-\mathbf{v}$, we have

$$
\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}(E(\mathbf{u}-\delta\mathbf{v})-E(\mathbf{u}))\\
&\quad=\int_{B_R\setminus\Gamma}\partial_j\sigma_{ij}v_i\,\mathrm{d}x-\int_{B_R\cap\Gamma}\sigma_{ij}^+n_j^+v_i^+\,\mathrm{d}\Gamma\\
&\qquad-\int_{B_R\cap\Gamma}\sigma_{ij}^-n_j^-v_i^-\,\mathrm{d}\Gamma-\int_{B_R\cap\Gamma}\partial_1W(u_1^+,u_2^+)v_1^++\partial_2W(u_1^+,u_2^+)v_2^+\geq 0.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&-\int_{B_R\setminus\Gamma}\partial_j\sigma_{ij}v_i\,\mathrm{d}x+\int_{B_R\cap\Gamma}\sigma_{ij}^+n_j^+v_i^+\,\mathrm{d}\Gamma\\
&\quad+\int_{B_R\cap\Gamma}\sigma_{ij}^-n_j^-v_i^-\,\mathrm{d}x\,\mathrm{d}z+\int_{B_R\cap\Gamma}\partial_1W(u_1^+,u_2^+)v_1^++\partial_2W(u_1^+,u_2^+)v_2^+\,\mathrm{d}\Gamma=0.
\end{aligned}
$$

Noticing that $\mathbf{n}^+=(0,0,-1)$ and $\mathbf{n}^-=(0,0,1)$, we have

$$
\begin{aligned}
&\int_{B_R\cap\Gamma}\sigma_{ij}^+n_j^+v_i^+\,\mathrm{d}\Gamma+\int_{B_R\cap\Gamma}\sigma_{ij}^-n_j^-v_i^-\,\mathrm{d}\Gamma\\
&\quad=\int_{B_R\cap\Gamma}-\sigma_{33}^+v_3^+\,\mathrm{d}x\,\mathrm{d}z+\int_{B_R\cap\Gamma}\sigma_{33}^-v_3^-\,\mathrm{d}\Gamma+\int_{B_R\cap\Gamma}-\sigma_{13}^+v_1^+\,\mathrm{d}\Gamma+\int_{B_R\cap\Gamma}\sigma_{13}^-v_1^-\,\mathrm{d}\Gamma\\
&\qquad+\int_{B_R\cap\Gamma}-\sigma_{23}^+v_2^+\,\mathrm{d}\Gamma+\int_{B_R\cap\Gamma}\sigma_{23}^-v_2^-\,\mathrm{d}\Gamma.
\end{aligned}
$$

Recall that $v_1^+=-v_1^-$, $v_3^+=v_3^-$ and $v_2^+=-v_2^-$. Hence due to the arbitrariness of R, we conclude that the minimizer $\mathbf{u}$ must satisfy

$$
\begin{aligned}
&\int_\Gamma\left[\sigma_{13}^++\sigma_{13}^--\partial_1W(u_1^+,u_2^+)\right]v_1^+\,\mathrm{d}\Gamma=0,\\
&\int_\Gamma\left[\sigma_{23}^++\sigma_{23}^--\partial_2W(u_1^+,u_2^+)\right]v_2^+\,\mathrm{d}\Gamma=0,\\
&\int_\Gamma\left(\sigma_{33}^+-\sigma_{33}^-\right)v_3^+\,\mathrm{d}\Gamma=0,\\
&\int_{\mathbb{R}^2\setminus\Gamma}(\nabla\cdot\sigma)\cdot\mathbf{v}\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z=0
\end{aligned}
$$

for any $\mathbf{v}\in C^\infty(B_R\setminus\Gamma)$ and $\mathbf{v}$ has compact support in B_R, which leads to the Euler–Lagrange equation (2.6). Here we write the equation $\nabla\cdot\sigma=0$ in $\mathbb{R}^2\setminus\Gamma$ as the first equation of (2.6) in terms of the displacement $\mathbf{u}$, using the constitutive relation. □

B Dirichlet to Neumann map

Proof of Lemma 2.2 Step 1. We take the Fourier transform of the elastic equations in (2.6) with respect to x_1, x_2 and denote the corresponding Fourier variables as k_1, k_2.

Due to (2.3), $\mathbf{u}$ is unbounded and we take the Fourier transform for $\mathbf{u}$ with respect to x_1, x_2 by regarding them as tempered distributions. For notation simplicity, denote the Fourier transforms to be $\hat{\mathbf{u}}$. Let $k=(k_1,k_2)$ and $|k|=\sqrt{k_1^2+k_2^2}$. We have

$$(1-2\nu)\partial_{33}\hat{u}_1 - [(2-2\nu)k_1^2 + (1-2\nu)k_2^2]\hat{u}_1 + ik_1\partial_3\hat{u}_3 - k_1k_2\hat{u}_2 = 0, \tag{B.1}$$

$$(2-2\nu)\partial_{33}\hat{u}_3 - (1-2\nu)|k|^2\hat{u}_3 + ik_1\partial_3\hat{u}_1 + ik_2\partial_3\hat{u}_2 = 0, \tag{B.2}$$

$$(1-2\nu)\partial_{33}\hat{u}_2 - [(2-2\nu)k_2^2 + (1-2\nu)k_1^2]\hat{u}_2 + ik_2\partial_3\hat{u}_3 - k_1k_2\hat{u}_1 = 0. \tag{B.3}$$

We can first eliminate $\hat{u}_2$ using (B.1), then eliminate $\hat{u}_3$ and obtain the ODE for $\hat{u}_1$

$$\partial_3^4\hat{u}_1 - 2|k|^2\partial_3^2\hat{u}_1 + |k|^4\hat{u}_1 = 0.$$

Next we use this ODE for $\hat{u}_1$ to simplify (B.1), (B.2), and (B.3) again and then eliminate $\hat{u}_1$ and $\hat{u}_2$ together. We obtain the ODE for $\hat{u}_3$

$$\partial_3^4\hat{u}_3 - 2|k|^2\partial_3^2\hat{u}_3 + |k|^4\hat{u}_3 = 0.$$

By the symmetry of $\hat{u}_1$ and $\hat{u}_2$, we have the same ODE for $\hat{u}_2$.

We look for solutions whose derivatives have decay properties, which exclude exponentially growing solutions as $|x_3| \to +\infty$. Denote

$$\hat{u}_1^- = (A^- + B^-|k|x_3)e^{|k|x_3}, \quad x_3 < 0,$$

where A^-, B^- are constants to be determined. Similarly, denote

$$\hat{u}_3^- = (C^- + D^-|k|x_3)e^{|k|x_3}, \quad \hat{u}_2^- = (E^- + F^-x_3|k|)e^{|k|x_3}, \quad x_3 < 0,$$

where C^-, D^-, E^-, F^- are constants to be determined. For $x_3 > 0$, we have another six constants A^+, B^+, C^+, D^+, E^+, F^+ to be determined and for $x_3 > 0$,

$$\hat{u}_1^+ = (A^+ - B^+|k|x_3)e^{-|k|x_3},$$
$$\hat{u}_3^+ = (C^+ - D^+|k|x_3)e^{-|k|x_3},$$
$$\hat{u}_2^+ = (E^+ - F^+|k|x_3)e^{-|k|x_3}.$$

Step 2. Given the Dirichlet values of u_1 and u_2, we express all the other constants by $A^\pm$ and $E^\pm$.

First, plugging $\hat{u}_1^-$, $\hat{u}_2^-$, and $\hat{u}_3^-$ into (B.1), we have

$$(2-4\nu)|k|^2B^- - k_1^2A^- + ik_1(C^-|k| + D^-|k|) - k_1k_2E^- = 0$$

and

$$-k_1^2B^- + ik_1D^-|k| - k_1k_2F^- = 0.$$

Plugging $\hat{u}_1^-$, $\hat{u}_2^-$, and $\hat{u}_3^-$ into (B.2), we have

$$|k|^2C^- + (4-4\nu)|k|^2D^- + ik_1|k|A^- + ik_1|k|B^- + ik_2|k|E^- + ik_2|k|F^- = 0$$

and

$$|k|^2D^- + ik_1|k|B^- + ik_2|k|F^- = 0.$$

Plugging $\hat{u}_1^-$, $\hat{u}_2^-$, and $\hat{u}_3^-$ into (B.3), we have

$$(2-4\nu)|k|^2F^- - k_2^2E^- + ik_2(C^-|k| + D^-|k|) - k_1k_2A^- = 0$$

and

$$-k_2^2F^- + ik_2D^-|k| - k_1k_2B^- = 0.$$

Simplifying these relations gives us

$$B^- = \frac{ik_1}{|k|} D^-, \quad F^- = \frac{ik_2}{|k|} D^-,$$
$$-k_1 A^- - k_2 E^- + i|k|C^- = (4\nu - 3) i |k| D^-.$$

Combining this with the boundary symmetry (2.2), we have

$$A^+ = -A^-, \quad B^+ = -B^-, \quad C^+ = C^-, \quad D^+ = D^-, \quad E^+ = -E^-, \quad F^+ = -F^-.$$

Then by $\sigma_{33}^+ = \sigma_{33}^-$ on Γ in (2.6), we further obtain $C^- = (2\nu - 1)D^-$. Therefore, all the other constants can be expressed in terms of A^- and E^-. In particular, we conclude that $\sigma_{13}(x_1, x_2, 0^+)$ and $\sigma_{23}(x_1, x_2, 0^+)$ can be expressed as in (2.7). □

References

1. Bass, R.F.: Regularity results for stable-like operators. J. Funct. Anal. **257**(8), 2693–2722 (2009)
2. Cabré, X., Sire, Y.: Nonlinear equations for fractional laplacians II: existence, uniqueness, and qualitative properties of solutions. Trans. Am. Math. Soc. **367**, 911–941 (2015)
3. Cabré, X., Cinti, E.: Energy estimates and 1-D symmetry for nonlinear equations involving the half-laplacian. Discrete Continuous Dyn. Syst. A **28**(3), 1179–1206 (2010)
4. Cabré, X., Cinti, E.: Sharp energy estimates for nonlinear fractional diffusion equations. Calc. Var. Partial. Differ. Equ. **49**(1–2), 233–269 (2014)
5. Cabré, X., Solà-Morales, J.: Layer solutions in a half-space for boundary reactions. Commun. Pure Appl. Math. **58**(12), 1678–1732 (2005)
6. Cinti, E., Serra, J., Valdinoci, E.: Quantitative flatness results and BV-estimates for stable nonlocal minimal surfaces. J. Differ. Geom. **112**(3), 447–504 (2019)
7. Dipierro, S., Serra, J., Valdinoci, E.: Nonlocal phase transitions: rigidity results and anisotropic geometry (2016). arXiv preprint arXiv:1611.03246
8. Dipierro, S., Serra, J., Valdinoci, E.: Improvement of flatness for nonlocal phase transitions. Am. J. Math. **142**(4), 1083–1160 (2020)
9. Dong, H., Kim, D.: Schauder estimates for a class of non-local elliptic equations. Discrete Continuous Dyn. Syst. A **33**(6), 2319–2347 (2013)
10. Figalli, A., Serra, J.: On stable solutions for boundary reactions: a De Giorgi-type result in dimension 4 + 1. Inventiones Math. **219**(1), 153–177 (2020)
11. Gao, Y., Liu, J.-G.: Long time behavior of dynamic solution to Peierls–Nabarro dislocation model. Methods Appl. Anal. **27**(2), 161–198 (2020)
12. Gao, Y., Liu, J.-G., Luo, T., Xiang, Y.: Revisit of the Peierls–Nabarro model for edge dislocations in hilbert space. To appear in Discrete & Continuous Dynamical Systems - B. https://doi.org/10.3934/dcdsb.2020224
13. Gui, C., Li, Q.: Some energy estimates for stable solutions to fractional Allen–Cahn equations. Calc. Var. Partial. Differ. Equ. **59**(2), 49 (2020)
14. Hirth, J.P., Lothe, J.: Theory of Dislocations, 2nd edn. Wiley, New York (1982)
15. Nabarro, F.R.N.: Dislocations in a simple cubic lattice. Proc. Phys. Soc. **59**, 256–272 (1947)
16. Palatucci, G., Savin, O., Valdinoci, E.: Local and global minimizers for a variational energy involving a fractional norm. Annali di Matematica **192**(4), 673–718 (2013)
17. Peierls, R.: The size of a dislocation. Proc. Phys. Soc. **52**, 34–37 (1940)
18. Savin, O.: Rigidity of minimizers in nonlocal phase transitions. Anal. PDE **11**(8), 1881–1900 (2018)
19. Xiang, Y., Wei, H., Ming, P., Weinan, E.: A generalized Peierls–Nabarro model for curved dislocations and core structures of dislocation loops in al and cu. Acta Mater. **56**, 1447–1460 (2008)
20. Xiang, Y.: Modeling dislocations at different scales. Commun. Comput. Phys. **1**(3), 383–424 (2006)