SIAM J. MATH. ANAL.
Vol. 58, No. 2, pp. 1017–1038

SLOW PATTERNS IN MULTILAYER DISLOCATION EVOLUTION WITH DYNAMIC BOUNDARY CONDITIONS*

YUAN GAO† AND STEFANIA PATRIZI‡

Abstract. In this paper, we study the slow patterns of multilayer dislocation dynamics modeled by a multiscale parabolic equation in the half-plane coupled with a dynamic boundary condition on the interface. We focus on the influence of bulk dynamics with various relaxation time scales, on the slow motion pattern on the interface governed by an ODE system. Starting from a superposition of N stationary transition layers, at a specific time scale for the interface dynamics, we prove that the dynamic solution approaches the superposition of N explicit transition profiles whose centers solve the ODE system with a repulsive force. Notably, this ODE system is identical to the one obtained in the slow motion patterns of the one-dimensional fractional Allen–Cahn equation, where the elastic bulk is assumed to be static. Due to the fully coupled bulk and interface dynamics, new corrector functions with delicate estimates are constructed to stabilize the bulk dynamics and characterize the limiting behavior of the dynamic solution throughout the entire half-plane.

Key words. reaction-diffusion, screw dislocation dynamics, interacting particle system, slow motion

MSC codes. M35B25-51, M35K20

DOI. 10.1137/25M1733136

1. Introduction.

In this paper, we study the slow patterns of dislocation dynamics modeled by a multiscale parabolic equation coupled with a dynamic boundary condition. Precisely, we aim to characterize the asymptotic behavior when $\varepsilon \to 0^+$ of the solution $u_\varepsilon = u_\varepsilon(x,y,t)$, $x \in \mathbb{R}$, $y \geq 0$, $t \geq 0$, to

$$\begin{cases} \varepsilon^a \partial_t u_\varepsilon - \Delta u_\varepsilon = 0, & y>0,\, t>0, \\ \varepsilon \partial_t u_\varepsilon - \partial_y u_\varepsilon + \dfrac{1}{\varepsilon} W'(u_\varepsilon) = 0, & y=0,\, t>0, \\ u_\varepsilon = u_\varepsilon^0, & t=0, \end{cases} \tag{1.1}$$

where $a > 0$ and W is a multiwell potential. In the following, we will introduce the background, specific setup, main results, and approaches.

1.1. Background and motivations.

Dislocations, which are line defects in crystalline materials, play a crucial role in the study of mechanical behaviors of materials. In particular, the motion of dislocations may cause permanent plastic deformations. To unveil the core structure of dislocations—small regions of heavily distorted

*Received by the editors February 11, 2025; accepted for publication (in revised form) October 27, 2025; published electronically March 11, 2026.
https://doi.org/10.1137/25M1733136
Funding: The first author was partially supported by NSF grant DMS-2204288 "Bulk-Interface Coupled Response in Novel Materials: Pattern Formation and Interactive Migration" and NSF grant DMS-2440651 "Fluctuation estimate, Selection principle and Transition paths for multiscale interacting dynamics on complex structure." The second author was supported by NSF grant DMS-2155156 "Nonlinear PDE methods in the study of interphases" and by Simons Foundation grant MPS-TSM-00697812.
†Department of Mathematics, Purdue University, West Lafayette, IN 47906 USA (gao662@purdue.edu).
‡Department of Mathematics, The University of Texas at Austin, Austin, TX 78751 USA (spatrizi@math.utexas.edu).

atomistic structures—the Peierls–Nabarro (PN) model, introduced by Peierls and Nabarro [Pei40, Nab47], is a multiscale continuum model for displacement $\mathbf{u}$. It incorporates the atomistic effect by introducing a nonlinear potential W that describes the atomistic misfit interaction across the dislocation's slip plane, while the elastic continua associated with elastic energy are connected by the interface misfit potential W.

Based on a simplified two-dimensional PN model, where the displacement $\mathbf{u}$ is replaced by a scalar variable u, our goal is to study the relaxation pattern of the dislocation dynamics with particular focus on the influence of the bulk dynamics with different time scaling. Precisely, consider the total free energy

$$E_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^2_+} \varepsilon|\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_\Gamma W(u)\,\mathrm{d}x. \tag{1.2}$$

Here the first term represents the elastic energy in the bulk $\mathbb{R}^2_+ := \mathbb{R}\times(0,\infty)$, and the second term represents the interface misfit energy on the slip plane $\Gamma := \{(x,y);\ x\in\mathbb{R}, y=0\}$. The rate of change of the total energy is determined by the dissipation functional. We consider the simplest quadratic dissipation functional including frictions in the bulks and on the interface:

$$g(\dot{u},\dot{v}) = \varepsilon^{a+1}\int_{\mathbb{R}^2_+} \dot{u}\dot{v}\,\mathrm{d}x\,\mathrm{d}y + \varepsilon^2\int_\Gamma \dot{u}\dot{v}\,\mathrm{d}x.$$

The gradient flow of $E_\varepsilon(u)$ with respect to the dissipation functional g is determined as follows. Suppose u and $\partial_t u$ are the true trajectory and the associated velocity. After perturbing the total energy in the direction of any virtual velocity $\dot{v}$ (cf. [Bow09, Chapter 2]), the weak form of gradient flow satisfies

$$\begin{aligned} g(\partial_t u,\dot{v}) &= \varepsilon^{a+1}\int_{\mathbb{R}^2_+} \partial_t u\dot{v}\,\mathrm{d}x\,\mathrm{d}y + \varepsilon^2\int_\Gamma \partial_t u\dot{v}\,\mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}\delta}\Big|_{\delta=0} E(u+\delta\dot{v})\\ &= -\int_{\mathbb{R}^2_+} \varepsilon\nabla u\nabla\dot{v}\,\mathrm{d}x\,\mathrm{d}y - \int_\Gamma W'(u)\dot{v}\,\mathrm{d}x \\ &= \int_{\mathbb{R}^2_+} \varepsilon\Delta u\dot{v}\,\mathrm{d}x\,\mathrm{d}y + \int_\Gamma [\varepsilon\partial_y u - W'(u)]\dot{v}\,\mathrm{d}x. \end{aligned} \tag{1.3}$$

By the arbitrariness of the virtual velocity $\dot{v}$, we obtain the governing equation (1.1).

Note that the dissipation scaling for the interface is fixed to be ε^2, as the slow motion behavior is only observed at this time scale for the dynamics on the interface [GM12]. On the other hand, the dissipation scaling for the bulk is ε^{a+1} with a parameter $a>0$. When a is large, the term ε^{a+1} indicates that the friction dissipation contributed from the bulk part is small and the bulk dynamics relax very fast toward equilibrium. Two singular cases are $a=0$ and $a\to+\infty$. If $a\to+\infty$, the bulk dynamics relax extremely fast to a stationary state, and thus it corresponds to the widely studied quasistatic bulk case $\Delta u = 0$. If $a=0$, then the bulk part is completely governed by a heat equation, for which one cannot expect the same slow motion behaviors of the layer solutions constructed by the stationary dislocation profile. We study the slow motion pattern of dislocation dynamics under various relaxation scaling with $a>0$ to include a wide range of materials with different stress-relaxation properties [SZdQ+22].

In particular, when $a \to +\infty$, the bulk dynamics relaxes very fast to a stationary state, and thus, by the Dirichlet-to-Neumann map for the Laplacian (see [CS07]), we have $\partial_y u(x,0) = -(-\Delta)^{\frac{1}{2}} u(x,0)$, and the full dynamics reduces to the classical one-dimensional fractional Allen–Cahn equation (a nonlocal reaction-diffusion equation):

$$\varepsilon \partial_t u_\varepsilon + (-\Delta)^{\frac{1}{2}} u_\varepsilon + \frac{1}{\varepsilon} W'(u_\varepsilon) = 0. \tag{1.4}$$

For (1.4) at this specific time scaling, the slow motion of a multilayer profile guided by an ODE system is studied by González and Monneau in [GM12]. This is also the reason we fixed the dissipation scaling on the interface to be ε^2. More precisely, in [GM12, Theorem 1.1] it is proven that the solution u_ε to (1.4), with a well-prepared initial datum—a superposition of N transition layers (see (1.8))—approaches as $\varepsilon \to 0^+$, and for every fixed time, the integers $1, 2, \ldots, N$, and the jump points, $z_i(t)$, between two consecutive integers, move accordingly to the following ODE system, for $i = 1, \ldots, N$:

$$\begin{cases} \dfrac{dz_i}{dt} = \dfrac{c_0}{\pi} \displaystyle\sum_{j \neq i} \frac{1}{z_i - z_j}, \quad t > 0, \\ z_i(0) = z_i^0. \end{cases} \tag{1.5}$$

Here z_i^0 is the center of each transition layer at initial time and $c_0 > 0$ is defined as in (2.1). The above slow motion pattern for multilayer profiles was previously studied for the classical one-dimensional Allen–Cahn equation [CP89, Che04]. In the nonlocal case, (1.4), where the operator $(-\Delta)^{\frac{1}{2}}$ is replaced by $(-\Delta)^s$, $s \in (0,1)$, and with the appropriate space scaling, is studied in [DFV14, DPV15]. In [PV15], more general initial data, including possible opposite orientations of dislocations, were considered and the slow motion pattern before a finite collision time was proven to be driven by a similar ODE system including either repulsive or attractive particle interactions. We refer the reader to [PV16, PV17] for comprehensive study of the long time behaviors of those multilayer dislocation profiles including possible finite time collisions. The case where $N \to \infty$ is studied in [PS21, PS23]. Properties of the ODE system (1.5) have been studied in [FIM09] and in the more general case in which collisions are allowed in [VMPP22].

Beyond the reduced one-dimensional nonlocal dynamics, the long time behavior of the fully coupled bulk-interface dynamics (1.1) with fixed $\varepsilon = 1$ and a single layer profile ($N = 1$) was established in [GR23]. Then the natural question is whether the same slow motion behavior of the multilayer dislocation profile, guided by the ODE system (1.5), observed in (1.4), can be obtained when the bulk dynamics, with different dissipation scalings, are coupled with the interface dynamics.

In this paper, we investigate the full dynamics (1.1), including the bulk dynamics with various relaxation scalings ε^a, and prove that the slow motion pattern driven by the ODE system (1.5) can still be observed, particularly for finite $a > 0$. The most delicate case is when $a \leq 1$, as discussed in subsection 1.3. Furthermore, we explicitly characterize the limiting behavior of the solution for all $y \geq 0$.

In the following subsection, we outline the specific setting and present our main result.

1.2. Setting of the problem and main result. Assume the nonconvex potential W satisfies

$$\begin{cases} W \in C^{2,\beta}(\mathbb{R}) & \text{for some } 0<\beta<1, \\ W(u+1)=W(u) & \text{for any } u\in\mathbb{R}, \\ W=0 & \text{on } \mathbb{Z}, \\ W>0 & \text{on } \mathbb{R}\setminus\mathbb{Z}, \\ W''(0)>0. \end{cases} \tag{1.6}$$

Let u_ε be the solution to (1.1) when the initial condition u_ε^0 is a superposition of layer solutions. The stationary layer solution (also called the phase transition) $\phi=\phi(x,y)$ is the unique solution to

$$\begin{cases} -\Delta\phi=0, & y>0, \\ \partial_y\phi=W'(\phi), & y=0, \\ \partial_x\phi>0 & \text{on } \mathbb{R}, \\ \phi(-\infty,y)=0, \quad \phi(+\infty,y)=1, \quad \phi(0,0)=\dfrac{1}{2}. \end{cases} \tag{1.7}$$

Then, for $z_1^0<z_2^0<\ldots<z_N^0$, we set initial data as

$$u_\varepsilon^0(x,y):=\sum_{i=1}^N \phi\left(\frac{x-z_i^0}{\varepsilon},\frac{y}{\varepsilon}\right). \tag{1.8}$$

When a special periodic misfit potential is chosen,

$$W'(u)=-\frac{1}{2\pi}\sin\left[2\pi\left(u-\frac{1}{2}\right)\right],$$

it is well known [CSM05, GM12] that

$$\Phi(x,y)=\frac{1}{\pi}\arctan\left(\frac{x}{y+1}\right)+\frac{1}{2} \tag{1.9}$$

is the only solution to (1.7). Further discussions on ϕ will be presented in section 2.2.

The following is the main result of our paper.

THEOREM 1.1. *Assume* (1.6) *and let u_ε be the solution to* (1.1) *with $a>0$. Assume the initial condition is given by* (1.8). *Let*

$$v_0(x,t):=\sum_{i=1}^N H(x-z_i(t)), \tag{1.10}$$

where H is the Heaviside function and $(z_1(t),\ldots,z_N(t))$ is the solution to (1.5). *Then, as $\varepsilon\to 0^+$, u_ε exhibits the following asymptotic behavior:*

(i) *For $t\ge 0$, $x\in\mathbb{R}$,*

$$\limsup_{\substack{(x',y',t')\to(x,0,t)\\ \varepsilon\to 0^+}} u_\varepsilon(x',y',t')\le (v_0)^*(x,t) \quad \text{and} \tag{1.11}$$
$$\liminf_{\substack{(x',y',t')\to(x,0,t)\\ \varepsilon\to 0^+}} u_\varepsilon(x',y',t')\ge (v_0)_*(x,t),$$

where $(v_0)_*$ and $(v_0)^*$ are the lower and upper semicontinuous envelopes of v_0.

(ii) *For $t\ge 0$, $x\in\mathbb{R}$, and $y>0$,*

$$\lim_{\varepsilon\to 0^+} u_\varepsilon(x,y,t)=\frac{1}{\pi}\sum_{i=1}^N\left(\frac{\pi}{2}+\arctan\left(\frac{x-z_i(t)}{y}\right)\right). \tag{1.12}$$

1.3. Heuristics. The main approach we use is to construct appropriate super/subsolutions to (1.1) and (1.8) as barrier functions. By applying the comparison principle, the limiting behavior of the dynamic solution u_ε is dominated by those barrier functions. These barriers are constructed from a formal ansatz that satisfies the equations and the initial condition, up to small errors.

For (1.4), the ansatz derived in [GM12] is given by

$$v_\varepsilon(x,y,t) := \sum_{i=1}^{N}\left[\phi\left(\frac{x-z_i(t)}{\varepsilon},\frac{y}{\varepsilon}\right) - \varepsilon\dot z_i(t)\psi\left(\frac{x-z_i(t)}{\varepsilon},\frac{y}{\varepsilon}\right)\right],$$

with $y=0$, where $(z_1(t),\dots,z_N(t))$ solves (1.5), ϕ is the layer solution given by (1.7), and ψ is a corrector that controls the error of order 1 when substituting the ansatz into (1.4). The equation for ψ is given in (2.6).

However, v_ε is not an appropriate ansatz for (1.1) when $y>0$. Indeed, using that ϕ and ψ are both harmonic for $y>0$, we obtain upon substituting v_ε into the equation

$$\varepsilon^a\partial_t v_\varepsilon - \Delta v_\varepsilon = -\varepsilon^{a-1}\sum_{i=1}^{N}\dot z_i(t)\partial_x\phi\left(\frac{x-z_i(t)}{\varepsilon},\frac{y}{\varepsilon}\right) + o_\varepsilon(1),$$

where $o_\varepsilon(1)\to 0$ as $\varepsilon\to 0^+$. Thus, the right-hand side of the equation is not small for $a\le 1$. To address this issue, we introduce an additional corrector q that satisfies

$$\begin{cases} -\Delta q(x,y) = \partial_x\phi(x,y), & (x,y)\in\mathbb{R}^2_+,\\ q(x,0)=0, & x\in\mathbb{R}. \end{cases} \tag{1.13}$$

We then consider the new ansatz w_ε which is obtained by adding a lower-order correction to v_ε,

$$w_\varepsilon(x,y,t) := v_\varepsilon(x,y,t) + \varepsilon^{a+1}\sum_{i=1}^{N}\dot z_i(t) q\left(\frac{x-z_i(t)}{\varepsilon},\frac{y}{\varepsilon}\right).$$

Formally, when we plug this ansatz into (1.1) for $y>0$, we obtain

$$\varepsilon^a\partial_t w_\varepsilon - \Delta w_\varepsilon = o_\varepsilon(1). \tag{1.14}$$

However, due to the lack of integrability properties in the entire half-plane, we must replace $\partial_x\phi(x,y)$ by $\partial_x\phi(x,y)g(y)$ in (1.13), where g is a cutoff function. Delicate growth estimates for the corrector q are crucial to control the bulk dynamics in the entire half-plane. These, along with new decay estimates for ϕ and ψ, are derived in section 2. To control the error in (1.14), additional terms must be added to w_ε; see section 3. Since in (1.1) the interface dynamics and bulk dynamics affect each other in a two-way coupling manner through the dynamic boundary condition with a Neumann derivative, these additional terms are introduced in a suitable way. Nevertheless, we are able to show that $w_\varepsilon - v_\varepsilon \to 0$ as $\varepsilon\to 0^+$, so that the limiting behavior of w_ε coincides with that of the harmonic limit of v_ε, which formally satisfies (1.1) with $\varepsilon=0$; see section 4.

1.4. Notation. In the paper, we will denote by $C>0$ any constant independent of ε.

For $\beta\in(0,1]$ and $k\in\mathbb{N}\cup\{0\}$, we denote by $C^{k,\beta}(\mathbb{R})$ the usual class of functions with bounded $C^{k,\beta}$ norm over $\mathbb{R}$.

We denote $\mathbb{R}^2_+ := \mathbb{R}\times(0,\infty)$.

Given a function $\eta=\eta(x,y,t)$, defined on a set A of $\overline{\mathbb{R}^2_+}\times[0,\infty)$, we write $\eta=O(\varepsilon)$ if there is $C>0$ such that $|\eta(x,y,t)|\le C\varepsilon$ for all $(x,y,t)\in A$.

For a set A, we denote by χ_A the characteristic function of A.

Given a function $v(x,y,t)$ we denote by v_* and v^* the lower and upper semicontinuous envelopes, respectively, defined by

$$v_*(x,y,t):=\liminf_{(x',y',t')\to(x,y,t)} v(x',y',t'),$$

$$v^*(x,y,t):=\limsup_{(x',y',t')\to(x,y,t)} v(x',y',t').$$

The Heaviside function is defined by

$$H(x):=\begin{cases}1 & \text{if } x>0,\\ 0 & \text{if } x<0.\end{cases}$$

The explicit value at 0, which is assumed to be in $[0,1]$, plays no role.

1.5. Organization of the paper. The rest of the paper is organized as follows. In section 2, we review and prove some preliminary results on stationary solutions and corrector functions. In section 3, we construct super- and subsolutions to the full dynamics (1.1) with initial condition (1.8). In section 4, we complete the proof of Theorem 1.1.

2. Preliminary results.

In this section, we will first review and establish some preliminary results. The properties of the ODE system (1.5) will be discussed in section 2.1, while the decay estimates for the stationary layer solution $\phi(x,y)$ will be presented in section 2.2. To prepare for the construction of super- and subsolutions in section 3, we introduce two corrector functions, $\psi(x,y)$, $q(x,y)$, and prove some essential decay estimates for each in sections 2.3 and 2.4, respectively.

2.1. Preliminary results on the ODE system. We first recall the following lower bound estimate for the minimal particle distance in ODE (1.5).

LEMMA 2.1 (see [VMPP22, Theorem 2.4]). *Let $(z_1(t),\dots,z_N(t))$ be the solution of* (1.5) *and let*

$$d(t):=\min\{|z_i(t)-z_j(t)|,\ i\neq j,\ i,j=1,\dots,N\}$$

be the minimal distance between dislocation points. Then

$$d(t)\ge\sqrt{\frac{8}{N^2-1}t+d(0)^2}.$$

2.2. The layer solution ϕ. Next, we summarize some properties of the layer function ϕ, solution to (1.7). For convenience in the notation, let c_0 and α be given, respectively, by

$$c_0^{-1}=\int_{\mathbb{R}}(\partial_x\phi(x,0))^2\,dx \quad\text{and}\quad \alpha=W''(0), \tag{2.1}$$

and let $H(x)$ be the Heaviside function.

LEMMA 2.2. *There is a unique solution* $\phi \in C^{2,\beta}(\overline{\mathbb{R}^2_+})$ *of* (1.7), *with the same* β *as in* (1.6). *Moreover, there exists a constant* $C>0$ *such that*

$$\left|\phi(x,0) - H(x) + \frac{1}{\alpha\pi x}\right| \le \frac{C}{x^2} \quad \text{for } |x| \ge 1 \tag{2.2}$$

and

$$\frac{y+1}{C(x^2+(y+1)^2)} \le \partial_x\phi(x,y) \le \frac{C(y+1)}{x^2+(y+1)^2} \quad \text{for all } (x,y) \in \overline{\mathbb{R}^2_+}. \tag{2.3}$$

In particular,

$$\frac{1}{Cx^2} \le \partial_x\phi(x,0) \le \frac{C}{x^2} \quad \text{for } |x| \ge 1. \tag{2.4}$$

Proof. Existence of a unique solution $\phi \in C^{2,\beta}(\overline{\mathbb{R}_2^+})$ of (1.7) is proven in [CSM05]; see Theorem 1.2 and Lemma 2.3. For estimate (2.3), see Theorem 1.6 and formulas (6.16) and (6.18) in the same paper. Estimate (2.2) is proven in [GM12, Theorem 3.1]. ∎

LEMMA 2.3. *Let* ϕ *be the solution of* (1.7), *given by Lemma* 2.2. *Then there exists* $C>0$ *such that for* ε, $y>0$,

$$\frac{1}{\pi}\left(\frac{\pi}{2} + \arctan\left(\frac{x-\varepsilon^{\frac12}}{y}\right)\right) - C\varepsilon^{\frac12} \le \phi\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right) \le \frac{1}{\pi}\left(\frac{\pi}{2} + \arctan\left(\frac{x+\varepsilon^{\frac12}}{y}\right)\right) + C\varepsilon^{\frac12}. \tag{2.5}$$

In particular, for $y>0$,

$$\lim_{\varepsilon\to 0^+} \phi\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right) = \frac{1}{\pi}\left(\frac{\pi}{2} + \arctan\left(\frac{x}{y}\right)\right).$$

Proof. Since ϕ is harmonic in $\mathbb{R}^2_+$, it can be written as the convolution of the Poisson kernel in the half-plane with $\phi(x,0)$. Thus for ε, $y>0$,

$$\phi\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right) = \frac{1}{\pi}\int_{\mathbb{R}} \phi(\xi,0)\,\frac{y/\varepsilon}{(x/\varepsilon-\xi)^2+(y/\varepsilon)^2}\,\mathrm{d}\xi = \frac{1}{\pi}\int_{\mathbb{R}} \phi\left(\frac{\zeta}{\varepsilon},0\right)\frac{y}{(x-\zeta)^2+y^2}\,\mathrm{d}\zeta,$$

where we performed the change of variable $\zeta=\varepsilon\xi$. By (2.2), and using that $\phi<1$, we obtain

$$\begin{aligned}
\frac{1}{\pi}\int_{\mathbb{R}} \phi\left(\frac{\zeta}{\varepsilon},0\right)\frac{y}{(x-\zeta)^2+y^2}\,\mathrm{d}\zeta &\le \frac{1}{\pi}\left(\int_{-\infty}^{-\varepsilon^{\frac12}} \frac{-C\varepsilon}{\zeta} + \int_{-\varepsilon^{\frac12}}^{\infty}\right)\frac{y}{(x-\zeta)^2+y^2}\,\mathrm{d}\zeta \\
&\le \frac{1}{\pi}\left(\int_{-\infty}^{-\varepsilon^{\frac12}} C\varepsilon^{\frac12} + \int_{-\varepsilon^{\frac12}}^{\infty}\right)\frac{y}{(x-\zeta)^2+y^2}\,\mathrm{d}\zeta \\
&\le C\varepsilon^{\frac12} + \frac{1}{\pi}\int_{-\varepsilon^{\frac12}}^{\infty} \frac{y}{(x-\zeta)^2+y^2}\,\mathrm{d}\zeta \\
&= \frac{1}{\pi}\left(\frac{\pi}{2} + \arctan\left(\frac{x+\varepsilon^{\frac12}}{y}\right)\right) + C\varepsilon^{\frac12}.
\end{aligned}$$

Therefore, we have

$$\phi\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\leq\frac{1}{\pi}\left(\frac{\pi}{2}+\arctan\left(\frac{x+\varepsilon^{\frac{1}{2}}}{y}\right)\right)+C\varepsilon^{\frac{1}{2}}.$$

Similarly, one can prove

$$\phi\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\geq\frac{1}{\pi}\left(\frac{\pi}{2}+\arctan\left(\frac{x-\varepsilon^{\frac{1}{2}}}{y}\right)\right)-C\varepsilon^{\frac{1}{2}}.$$

Estimate (2.5) follows. ∎

2.3. The corrector ψ. We now introduce the first corrector ψ, which will be used later to control the interface dynamics. As in [GM12], we define the function ψ to be the solution of

$$\begin{cases}-\Delta\psi=0, & y>0,\\ \partial_y\psi=W''(\phi)\psi+\dfrac{1}{\alpha c_0}(W''(\phi)-W''(0))+\partial_x\phi, & y=0,\\ \lim_{|x|\to\infty}\psi(x,0)=0,\end{cases}\tag{2.6}$$

where c_0, α are defined as in (2.1). We will use ψ as an $O(\varepsilon)$ correction to construct sub- and supersolutions to (1.1). For a detailed heuristic motivation of (2.6), see [GM12, section 3.1]. In the next lemma, we present some known results about the function ψ as well as new estimates.

LEMMA 2.4. *There exists a unique solution $\psi\in C^{1,\beta}_{loc}(\overline{\mathbb{R}^2_+})\cap W^{1,\infty}(\overline{\mathbb{R}^2_+})$ to (2.6). Furthermore, there exist constants $c\in\mathbb{R}$ and $C>0$ such that*

$$\left|\psi(x,0)-\frac{c}{x}\right|\leq\frac{C}{x^2},\quad|\partial_x\psi(x,0)|\leq\frac{C}{x^2}\quad\textit{for } |x|\geq1,\tag{2.7}$$

$$|\psi(x,y)|\leq\frac{C}{y},\quad|\partial_x\psi(x,y)|\leq\frac{C}{y}\quad\textit{for all } x\in\mathbb{R} \textit{ and } y\geq1,\tag{2.8}$$

and

$$|\psi(x,y)|\leq\frac{C}{|x|}\quad\textit{for all } x>1 \textit{ and } y\geq0.\tag{2.9}$$

Proof. Existence of a solution $\psi\in C^{1,\beta}_{loc}(\overline{\mathbb{R}^2_+})\cap W^{1,\infty}(\overline{\mathbb{R}^2_+})$ of (2.6) is proven in [GM12, Theorem 3.2]. Estimates (2.7) are proven in [MP12, Lemma 3.2].

Let us prove (2.8). Let Φ be the explicit layer solution given by (1.9). Then Φ satisfies (2.2) with $\alpha=1$. In particular, for a, $b>0$,

$$\Phi\left(\frac{x}{b},0\right)-\Phi\left(\frac{x}{a},0\right)\leq\frac{a-b}{\pi x}+\frac{C}{x^2+1}.$$

Choosing a and b such that $b-a=c$ with c as in (2.7), we see that

$$\psi(x,0)\leq\Phi\left(\frac{x}{a},0\right)-\Phi\left(\frac{x}{b},0\right)+\frac{C}{x^2+1}\leq\Phi\left(\frac{x}{a},0\right)-\Phi\left(\frac{x}{b},0\right)+C\partial_x\Phi(x,0).$$

Since the functions ψ, Φ and $\partial_x\Phi$ are all harmonic in $\mathbb{R}^2_+$, the comparison principle implies that, for all $(x,y)\in\overline{\mathbb{R}^2_+}$,

$$\psi(x,y)\le \Phi\left(\frac{x}{a},\frac{y}{a}\right)-\Phi\left(\frac{x}{b},\frac{y}{b}\right)+C\partial_x\Phi(x,y). \tag{2.10}$$

Now, by the mean value theorem, for some $\lambda\in(0,1)$,

$$\begin{aligned}\Phi\left(\frac{x}{a},\frac{y}{a}\right)-\Phi\left(\frac{x}{b},\frac{y}{b}\right)&=\frac{1}{\pi}\arctan\left(\frac{x}{y+a}\right)-\frac{1}{\pi}\arctan\left(\frac{x}{y+b}\right)\\&=\frac{1}{\pi}\frac{1}{1+s^2}_{|s=x\left(\frac{\lambda}{y+a}+\frac{1-\lambda}{y+b}\right)}x\left(\frac{1}{y+a}-\frac{1}{y+b}\right)\\&=\frac{b-a}{\pi}\frac{(y+a)(y+b)x}{(y+a)^2(y+b)^2+x^2(y+\lambda b+(1-\lambda)a)^2}.\end{aligned}$$

If $1\le y\le 2\max\{a,b\}$, then

$$\Phi\left(\frac{x}{a},\frac{y}{a}\right)-\Phi\left(\frac{x}{b},\frac{y}{b}\right)\le C\le\frac{\tilde C}{y}.$$

If instead $y>2\max\{a,b\}$, then

$$\Phi\left(\frac{x}{a},\frac{y}{a}\right)-\Phi\left(\frac{x}{b},\frac{y}{b}\right)\le\frac{Cy^2|x|}{y^4+x^2y^2}=\frac{C|x|}{y^2+x^2}\le\frac{\tilde C}{y}.$$

Combining the last two inequalities with (2.10) yields

$$\psi(x,y)\le\frac{C}{y}\quad\text{for } y\ge 1.$$

Similarly, one can prove

$$\psi(x,y)\ge-\frac{C}{y}\quad\text{for } y\ge 1.$$

Estimate (2.8) for ψ follows. The same argument also gives estimate (2.9).

Next, from (2.4) and (2.7), there is $C>0$ such that $-C\partial_x\phi(x,0)\le\partial_x\psi(x,0)\le C\partial_x\phi(x,0)$. Since both $\partial_x\phi$ and $\partial_x\psi$ are harmonic in $\mathbb{R}^2_+$, by the comparison principle we get $-C\partial_x\phi(x,y)\le\partial_x\psi(x,y)\le C\partial_x\phi(x,y)$ for every $(x,y)\in\overline{\mathbb{R}^2_+}$, which combined with (2.3) gives (2.8) for $\partial_x\psi$. ◻

2.4. The corrector *q*. We introduce a further corrector, q, solution to

$$\begin{cases}-\Delta q(x,y)=\partial_x\phi(x,y)g(y), & (x,y)\in\mathbb{R}^2_+,\\ q(x,0)=0, & x\in\mathbb{R},\end{cases} \tag{2.11}$$

where g is a smooth cut-off function. We will use q as an $O(\varepsilon^{a+1})$ correction to control the bulk dynamics when constructing sub- and supersolutions to (1.1).

Existence and properties of q are proven in the next lemma.

LEMMA 2.5. *Let $g(y)$ be a smooth nonnegative function with support in $[0,R]$, $R>2$. Then there exists a unique bounded solution q of* (2.11)*, where ϕ is the solution of* (1.7)*. Moreover, there exists a constant $C>0$, such that*

$$0 \le q(x,y) \le CR\ln R \quad \textit{for all } (x,y) \in \overline{\mathbb{R}^2_+}, \tag{2.12}$$

$$|\partial_x q(x,y)|,\, |\partial_y q(x,y)| \le C\ln R \quad \textit{for all } (x,y) \in \overline{\mathbb{R}^2_+}, \tag{2.13}$$

and

$$q(x,y) \le \frac{CR^2}{y}, \quad |\partial_x q(x,y)|,\, |\partial_y q(x,y)| \le \frac{CR}{y} \quad \textit{for all } x \in \mathbb{R} \textit{ and } y \ge 2R. \tag{2.14}$$

Proof. Consider the Green function in the half-plane, given by

$$G(Z', Z) = \frac{1}{2\pi}\left(\ln|Z' - \tilde{Z}| - \ln|Z' - Z|\right),$$

where if $Z = (x,y) \in \mathbb{R}^2_+$, then $\tilde{Z} = (x, -y)$. Define

$$f(Z) := \partial_x \phi(Z) g(y)$$

and

$$q(Z) := \int_{\mathbb{R}^2_+} G(Z', Z) f(Z')\, \mathrm{d}Z'.$$

We will show that q is well-defined and satisfies estimates (2.12)–(2.14). In particular, q is a smooth solution of (2.11). The uniqueness is a consequence of the uniqueness of the bounded solution to (2.11).

Since for Z, $Z' \in \mathbb{R}^2_+$,

$$|Z' - \tilde{Z}| \le |Z' - Z| + |Z - \tilde{Z}| = |Z' - Z| + 2y,$$

we have that

(2.15)
$$0 \le \ln|Z' - \tilde{Z}| - \ln|Z' - Z| \le \ln(|Z' - Z| + 2y) - \ln(|Z' - Z|) = \ln\left(1 + \frac{2y}{|Z' - Z|}\right).$$

Moreover, by (2.3),

$$0 \le f(Z') \le \frac{C(y'+1)}{x'^2 + (y'+1)^2}\chi_{[0,R]}(y') \le \frac{C}{y'+1}\chi_{[0,R]}(y'). \tag{2.16}$$

Since both f and G are nonnegative, nonzero functions, q is positive. Let us show the upper bound for q in (2.12). In view of (2.14), we may assume that $y < 2R$. We write

$$q(Z) = \left(\int_{\mathbb{R}^2_+ \cap \{|Z'-Z|<1\}} + \int_{\mathbb{R}^2_+ \cap \{|Z'-Z|>1\}}\right) G(Z', Z) f(Z')\, \mathrm{d}Z' =: I_1 + I_2. \tag{2.17}$$

By (2.15), and using that $\ln|Z' - Z|$ is integrable in $\{|Z' - Z| < 1\}$, we have

$$\begin{aligned} I_1 &\le \frac{1}{2\pi}\int_{\mathbb{R}^2_+ \cap \{|Z'-Z|<1\}} (\ln(1+2y) - \ln|Z' - Z|) f(Z')\, \mathrm{d}Z' \\ &\le \frac{\ln(1+2y)}{2\pi}\int_{\mathbb{R}^2_+ \cap \{|Z'-Z|<1\}} f(Z')\, \mathrm{d}Z' + C. \end{aligned}$$

Since $y < 2R$, we get

$$I_1 \le C \ln R. \tag{2.18}$$

Next, by (2.15) and the fact that g has support in $[0,R]$, we have

$$I_2 \le C\ln(1+2y)\int_0^R \mathrm{d}y' \int_{\mathbb{R}} \partial_x \phi(x',y')\,\mathrm{d}x' \le CR\ln(1+2y) \le CR\ln R, \tag{2.19}$$

where we used again that $y<2R$, and that $\phi(\infty,y')=1$, $\phi(-\infty,y')=0$.

From (2.17), (2.18), and (2.19), we obtain estimate (2.12).

Next, we compute

$$\begin{aligned} \partial_y G(Z',Z) &= \frac{1}{2\pi}\left(\frac{y'+y}{|Z'-\tilde{Z}|^2} + \frac{y'-y}{|Z'-Z|^2}\right),\\ \partial_x G(Z',Z) &= \frac{1}{2\pi}\left(\frac{x-x'}{|Z'-\tilde{Z}|^2} - \frac{x-x'}{|Z'-Z|^2}\right). \end{aligned} \tag{2.20}$$

Therefore, using (2.16), we get

$$\begin{aligned} |\partial_y q(Z)| &= \left|\int_{\mathbb{R}^2_+} \partial_y G(Z',Z) f(Z')\,\mathrm{d}Z'\right|\\ &\le C\int_{\mathbb{R}^2_+\cap\{0<y'<R\}} \left(\frac{y'+y}{(x'-x)^2+(y'+y)^2} + \frac{|y'-y|}{(x'-x)^2+(y'-y)^2}\right)\frac{1}{y'+1}\,\mathrm{d}x'\,\mathrm{d}y'\\ &= C\int_0^R \mathrm{d}y' \,\frac{1}{y'+1}\int_{\mathbb{R}}\left(\frac{y'+y}{(x'-x)^2+(y'+y)^2} + \frac{|y'-y|}{(x'-x)^2+(y'-y)^2}\right)\mathrm{d}x'\\ &= C\int_0^R \frac{1}{y'+1}\left(\arctan\left(\frac{x'-x}{y'+y}\right)\Bigg|_{x'=-\infty}^{x'=\infty} + \arctan\left(\frac{x'-x}{|y'-y|}\right)\Bigg|_{x'=-\infty}^{x'=\infty}\right)\mathrm{d}y'\\ &= C\int_0^R \frac{1}{y'+1}\,\mathrm{d}y' \le C\ln R, \end{aligned}$$

which proves (2.13) for $\partial_y q$.

To estimate $\partial_x q$, using (2.16) and performing an integration by parts we get

$$\begin{aligned} &|\partial_x q(Z)|\\ &= \left|\int_{\mathbb{R}^2_+} \partial_x G(Z',Z) f(Z')\,\mathrm{d}Z'\right|\\ &\le C\int_{\mathbb{R}^2_+\cap\{0<y'<R\}} \left(\frac{|x'-x|}{(x'-x)^2+(y'+y)^2} + \frac{|x'-x|}{(x'-x)^2+(y'-y)^2}\right)\frac{y'+1}{x'^2+(y'+1)^2}\,\mathrm{d}x'\,\mathrm{d}y'\\ &= C\int_{\mathbb{R}} \mathrm{d}x' \int_0^R \left(\frac{|x'-x|}{(x'-x)^2+(y'+y)^2} + \frac{|x'-x|}{(x'-x)^2+(y'-y)^2}\right)\frac{y'+1}{x'^2+(y'+1)^2}\,\mathrm{d}y'\\ &= -C\int_{\mathbb{R}} \mathrm{d}x' \int_0^R \left(\arctan\left(\frac{y'+y}{|x'-x|}\right) + \arctan\left(\frac{y'-y}{|x'-x|}\right)\right)\partial_{y'}\left(\frac{y'+1}{x'^2+(y'+1)^2}\right)\mathrm{d}y'\\ &\quad + C\int_{\mathbb{R}}\left(\arctan\left(\frac{y'+y}{|x'-x|}\right) + \arctan\left(\frac{y'-y}{|x'-x|}\right)\right)\frac{y'+1}{x'^2+(y'+1)^2}\Bigg|_{y'=0}^{y'=R}\mathrm{d}x'\\ &\le C\int_0^R \mathrm{d}y' \int_{\mathbb{R}} \frac{1}{x'^2+(y'+1)^2}\,\mathrm{d}x' + C\int_{\mathbb{R}}\left(\frac{R+1}{x'^2+(R+1)^2} + \frac{1}{x'^2+1}\right)\mathrm{d}x' \end{aligned}$$

$$= C\int_0^R \frac{1}{y'+1}\arctan\left(\frac{x'}{y'+1}\right)\Bigg|_{x'=-\infty}^{x'=\infty} \mathrm{d}y' + C\left(\arctan\left(\frac{x'}{R+1}\right)+\arctan x'\right)\Bigg|_{x'=-\infty}^{x'=\infty}$$
$$= C\int_0^R \frac{\mathrm{d}y'}{1+y'} + C \le C\ln R.$$

Estimate (2.13) for $\partial_x q$ is then proven.

Finally, assume $y > 2R$. Then, for $0 < y' < R$, we have that

$$|Z - Z'| \ge y - y' \ge \frac{y}{2}, \tag{2.21}$$

from which

$$\begin{aligned} G(Z', Z) &= \frac{1}{4\pi}\left[\ln((x'-x)^2+(y'+y)^2) - \ln((x'-x)^2+(y'-y)^2)\right] \\ &= \frac{1}{4\pi}\int_{(y'-y)^2}^{(y'+y)^2} \frac{\mathrm{d}\tau}{(x'-x)^2+\tau} \\ &\le \frac{1}{\pi}\frac{y'y}{(x'-x)^2+(y'-y)^2} = \frac{1}{\pi}\frac{y'y}{|Z'-Z|^2} \le \frac{1}{\pi}\frac{Ry}{|Z'-Z|^2} \le \frac{4R}{\pi y}. \end{aligned}$$

Therefore, we get

$$q(Z) \le \frac{CR}{y}\int_0^R \mathrm{d}y' \int_{\mathbb{R}} \partial_x\phi(x', y')\,\mathrm{d}x' \le \frac{CR^2}{y},$$

where we used again that $\phi(\infty, y') = 1$, $\phi(-\infty, y') = 0$. This gives (2.14) for q.

Next, recalling (2.20), by (2.21), for $y > 2R$ and $0 < y' < R$,

$$|\partial_y G(Z', Z)| \le \frac{1}{\pi|Z'-Z|} \le \frac{2}{\pi y}.$$

Therefore,

$$|\partial_y q(Z)| \le \frac{C}{y}\int_0^R \mathrm{d}y' \int_{\mathbb{R}} \partial_x\phi(x', y')\,\mathrm{d}x' \le \frac{CR}{y},$$

which proves (2.14) for $\partial_y q$. The estimate for $\partial_x q$ follows similarly. The proof of the lemma is then complete. ∎

3. Construction of super- and subsolutions to (1.1) with (1.8). In this section, we construct a supersolution w_ε and a subsolution h_ε to (1.1) with initial condition (1.8) based on multilayers of transition profiles and appropriate correctors, whose centers solve slightly perturbed ODE systems. In Propositions 3.2 and 3.3, we will prove $w_\varepsilon(x, y, t)$ is a supersolution to (1.1) with initial datum (1.8). The subsolution result will be summarized in Proposition 3.4.

Consider the perturbed ODE system, for $i = 1, \ldots, N$, $\delta > 0$,

$$\begin{cases} \dfrac{d\bar{z}_i}{dt} = \dfrac{c_0}{\pi}\left(\displaystyle\sum_{j\neq i}\frac{1}{\bar{z}_i - \bar{z}_j} - \delta\right), \quad t > 0, \\ \bar{z}_i(0) = z_i^0 - \delta, \end{cases} \tag{3.1}$$

and let

$$\bar{c}_i(t) := \dot{\bar{z}}_i(t), \quad \tilde{\delta} = \frac{\delta}{\alpha}, \tag{3.2}$$

with α defined as in (2.1). Define

$$v_\varepsilon(x,y,t) := \sum_{i=1}^{N}\left[\phi\left(\frac{x-\bar z_i(t)}{\varepsilon},\frac{y}{\varepsilon}\right) - \varepsilon\bar c_i(t)\psi\left(\frac{x-\bar z_i(t)}{\varepsilon},\frac{y}{\varepsilon}\right)\right] + \varepsilon\tilde\delta. \tag{3.3}$$

Lemma 3.1 (see [GM12, Proposition 4.3]). *There exist ε_0, $\delta_0>0$ such that, for any $\varepsilon<\varepsilon_0$, if $(\bar z_1(t),\dots,\bar z_N(t))$ is the solution of* (3.1) *with $0<\delta<\delta_0$, then the function v_ε defined in* (3.3) *solves*

$$\begin{cases} -\Delta v_\varepsilon = 0, & y>0,\\ \varepsilon\partial_t v_\varepsilon - \partial_y v_\varepsilon + \dfrac{1}{\varepsilon}W'(v_\varepsilon) \ge \dfrac{\delta}{2}, & y=0. \end{cases}$$

We now introduce a new function w_ε which is obtained by adding a correction to the function v_ε. More precisely, let q be given by Lemma 2.5, where we choose $R=2\varepsilon^{-b}$, with $0<b<1$, and g to be a smooth nonnegative cut-off function such that

$$g(y) = \begin{cases} 1 & \text{for } 0\le y\le \frac{R}{2}=\varepsilon^{-b},\\ 0 & \text{for } y\ge R=2\varepsilon^{-b}. \end{cases} \tag{3.4}$$

For $\tau>0$, $\theta>0$, and $0<\gamma<1$, define

$$w_\varepsilon(x,y,t) := v_\varepsilon(x,y,t) + \varepsilon^{a+1}\sum_{i=1}^{N}\bar c_i(t)q\left(\frac{x-\bar z_i(t)}{\varepsilon},\frac{y}{\varepsilon}\right) + \varepsilon^\theta(y+\varepsilon)^\gamma + \varepsilon^{1+\tau}t. \tag{3.5}$$

Proposition 3.2. *Given $T>0$ and $a>0$, there exist ε_0, δ_0, τ, $\theta>0$ and $0<b,\gamma<1$ such that, for any $0<\varepsilon<\varepsilon_0$, if $(\bar z_1(t),\dots,\bar z_N(t))$ is the solution of* (3.1) *with $0<\delta<\delta_0$, then the function w_ε defined in* (3.5) *solves*

$$\begin{cases} \varepsilon^a\partial_t w_\varepsilon - \Delta w_\varepsilon \ge 0, & y>0,\ t\in(0,T),\\ \varepsilon\partial_t w_\varepsilon - \partial_y w_\varepsilon + \dfrac{1}{\varepsilon}W'(w_\varepsilon)\ge 0, & y=0,\ t\in(0,T). \end{cases} \tag{3.6}$$

Proof. For convenience, we use the following notation throughout the proof:

$$h_i := h\left(\frac{x-\bar z_i(t)}{\varepsilon},\frac{y}{\varepsilon}\right)$$

with $h=\phi$, $\partial_x\phi$, $\partial_y\phi$, ψ, $\partial_x\psi$, $\partial_y\psi$, q, $\partial_x q$, $\partial_y q$.

Let $0<\varepsilon<\varepsilon_0$ and $0<\delta<\delta_0$ with ε_0 and δ_0 given by Lemma 3.1. Note that, by Lemma 2.1, for δ_0 small enough, we have that $\bar c_i$ and $\dot{\bar c}_i$ are bounded in $[0,T]$.

At $y=0$, recalling that $q(x,0)=0$, we also have $\partial_x q(x,0)=0$, and therefore

$$\begin{aligned} &\varepsilon\partial_t w_\varepsilon - \partial_y w_\varepsilon + \frac{1}{\varepsilon}W'(w_\varepsilon)\\ &\quad= \varepsilon\partial_t v_\varepsilon + \varepsilon^{2+\tau} - \partial_y v_\varepsilon - \varepsilon^a\sum_{i=1}^{N}\bar c_i\partial_y q_i - \gamma\varepsilon^{\theta+\gamma-1} + \frac{1}{\varepsilon}W'(v_\varepsilon + \varepsilon^{\theta+\gamma} + \varepsilon^{1+\tau}t)\\ &\quad= \varepsilon\partial_t v_\varepsilon - \partial_y v_\varepsilon + \frac{1}{\varepsilon}W'(v_\varepsilon) - \varepsilon^a\sum_{i=1}^{N}\bar c_i\partial_y q_i + O(\varepsilon^{\theta+\gamma-1}) + O(\varepsilon^\tau T). \end{aligned}$$

By Lemma 3.1 and (2.13) with $R=2\varepsilon^{-b}$, and by eventually making ε_0 smaller, we get, for $\varepsilon<\varepsilon_0$,

$$\varepsilon\partial_t w_\varepsilon - \partial_y w_\varepsilon + \frac{1}{\varepsilon}W'(w_\varepsilon) \geq \frac{\delta}{2} + O(\varepsilon^a|\ln\varepsilon|) + O(\varepsilon^{\theta+\gamma-1}) + O(\varepsilon^\tau T) \geq 0,$$

provided

$$\theta + \gamma > 1. \tag{3.7}$$

For $y > 0$, using that ϕ, ψ, and q satisfy, respectively, (1.7), (2.6), and (2.11), we get

$$\begin{aligned}\varepsilon^a\partial_t w_\varepsilon - \Delta w_\varepsilon = &-\varepsilon^{a-1}\sum_{i=1}^N \bar{c}_i(t)\partial_x\phi_i + \varepsilon^a\sum_{i=1}^N \bar{c}_i^2(t)\partial_x\psi_i - \varepsilon^{a+1}\sum_{i=1}^N \dot{\bar{c}}_i(t)\psi_i \\ &- \varepsilon^{2a}\sum_{i=1}^N \bar{c}_i^2(t)\partial_x q_i + \varepsilon^{2a+1}\sum_{i=1}^N \dot{\bar{c}}_i(t)q_i + \varepsilon^{a-1}g\sum_{i=1}^N \bar{c}_i(t)\partial_x\phi_i \\ &+ \varepsilon^{a+1+\tau} + \gamma(1-\gamma)\varepsilon^\theta(y+\varepsilon)^{\gamma-2}.\end{aligned} \tag{3.8}$$

The proof of (3.6) for $y > 0$ is broken into four cases.

Case 1: $0 < y \leq \frac{\varepsilon R}{2} = \varepsilon^{1-b}$. In this first case, by (3.4) $g = g(\frac{y}{\varepsilon}) = 1$, so that (3.8) reads

$$\begin{aligned}\varepsilon^a\partial_t w_\varepsilon - \Delta w_\varepsilon = &\varepsilon^a\sum_{i=1}^N \bar{c}_i^2(t)\partial_x\psi_i - \varepsilon^{a+1}\sum_{i=1}^N \dot{\bar{c}}_i(t)\psi_i - \varepsilon^{2a}\sum_{i=1}^N \bar{c}_i^2(t)\partial_x q_i \\ &+ \varepsilon^{2a+1}\sum_{i=1}^N \dot{\bar{c}}_i(t)q_i + \varepsilon^{1+a+\tau} + \gamma(1-\gamma)\varepsilon^\theta(y+\varepsilon)^{\gamma-2}.\end{aligned}$$

By (2.12) and (2.13) with $R = 2\varepsilon^{-b}$, we have

$$0 \leq q_i \leq C\varepsilon^{-b}|\ln\varepsilon| \quad \text{and} \quad |\partial_x q_i| \leq C|\ln\varepsilon|, \tag{3.9}$$

from which, recalling that $b < 1$,

$$\begin{aligned}\varepsilon^a\partial_t w_\varepsilon - \Delta w_\varepsilon &\geq O(\varepsilon^a) + O(\varepsilon^{2a}|\ln\varepsilon|) + O(\varepsilon^{2a+1-b}|\ln\varepsilon|) + \gamma(1-\gamma)\varepsilon^\theta(y+\varepsilon)^{\gamma-2} \\ &\geq O(\varepsilon^a) + \gamma(1-\gamma)\varepsilon^\theta \geq 0\end{aligned}$$

for ε small enough, provided

$$0 < \theta < a. \tag{3.10}$$

Next, for any given $b < 1$, let k_0 be the first integer such that

$$1 - (k_0+1)b \leq 0. \tag{3.11}$$

Notice that $k_0 \geq 1$.

Case 2: $\varepsilon^{1-kb} \leq y \leq \varepsilon^{1-(k+1)b}$, $k = 1, \ldots, k_0$. By (2.3) and (2.8),

$$0 \leq \partial_x\phi_i,\ |\psi_i|,\ |\partial_x\psi_i| \leq \frac{C\varepsilon}{y} \leq C\varepsilon^{kb}.$$

Combining these estimates with (3.9), and using that, since $1 - kb > 0$,

$$2a > a - (1-kb) \quad \text{and} \quad 2a + 1 - b > a - (1-kb),$$

(3.8) can be estimated as

$$\begin{aligned}\varepsilon^a \partial_t w_\varepsilon - \Delta w_\varepsilon &\geq O(\varepsilon^{a+kb-1}) + O(\varepsilon^{2a}|\ln\varepsilon|) + O(\varepsilon^{2a+1-b}|\ln\varepsilon|)\\ &\quad + \gamma(1-\gamma)\varepsilon^\theta (y+\varepsilon)^{\gamma-2}\\ &= O(\varepsilon^{a+kb-1}) + \gamma(1-\gamma)\varepsilon^\theta(y+\varepsilon)^{\gamma-2}\\ &\geq O(\varepsilon^{a+kb-1}) + C\varepsilon^{\theta-(2-\gamma)[1-(k+1)b]} \geq 0\end{aligned}$$

for ε small enough, provided

$$a + kb - 1 > \theta - (2-\gamma)[1-(k+1)b]. \tag{3.12}$$

Next, let k_1 be the first integer such that

$$\frac{k_1+1}{2}a > 1. \tag{3.13}$$

Notice that $k_1 \geq 0$.

Case 3: $\varepsilon^{-\frac{k}{2}a} \leq y \leq \varepsilon^{-\frac{k+1}{2}a}$, $k = 0, \dots, k_1$. In this case, by (2.3) and (2.8),

$$0 \leq \partial_x \phi_i,\, |\psi_i|,\, |\partial_x \psi_i| \leq \frac{C\varepsilon}{y} \leq C\varepsilon^{\frac{k}{2}a+1},$$

and by (2.14) with $R = 2\varepsilon^{-b}$,

$$|q_i| \leq C\frac{R^2\varepsilon}{y} \leq C\varepsilon^{\frac{k}{2}a+1-2b}, \quad |\partial_x q_i| \leq C\frac{R\varepsilon}{y} \leq C\varepsilon^{\frac{k}{2}a+1-b}.$$

Therefore, by (3.8), and using that $b<1$, we have

$$\begin{aligned}\varepsilon^a \partial_t w_\varepsilon - \Delta w_\varepsilon &\geq O\left(\varepsilon^{\left(1+\frac{k}{2}\right)a}\right) + O\left(\varepsilon^{\left(2+\frac{k}{2}\right)a+1-b}\right) + O\left(\varepsilon^{\left(2+\frac{k}{2}\right)a+2(1-b)}\right)\\ &\quad + \gamma(1-\gamma)\varepsilon^\theta(y+\varepsilon)^{\gamma-2}\\ &= O\left(\varepsilon^{\left(1+\frac{k}{2}\right)a}\right) + \gamma(1-\gamma)\varepsilon^\theta(y+\varepsilon)^{\gamma-2}\\ &\geq O\left(\varepsilon^{\left(1+\frac{k}{2}\right)a}\right) + C\varepsilon^{\theta+(2-\gamma)\frac{k+1}{2}a} \geq 0\end{aligned}$$

for ε small enough, provided

$$\left(1+\frac{k}{2}\right)a > \theta + (2-\gamma)\frac{k+1}{2}a. \tag{3.14}$$

Next, by (3.13) there exists $r>0$ so small that

$$\frac{k_1+1}{2}a > 1 + r. \tag{3.15}$$

Then we are left with the following last case.

Case 4: $y \geq \varepsilon^{-1-r}$. In this case, by (2.3) and (2.8),

$$0 \leq \partial_x \phi_i,\, |\psi_i|,\, |\partial_x \psi_i| \leq \frac{C\varepsilon}{y} \leq C\varepsilon^{2+r},$$

and by (2.14),

$$|q_i| \leq \frac{CR^2\varepsilon}{y} \leq C\varepsilon^{2+r-2b}, \quad |\partial_x q_i| \leq \frac{CR\varepsilon}{y} \leq \varepsilon^{2+r-b}.$$

Therefore, by (3.8) and using that for $b<1$,

$$2a-b+2+r,\, 2(a-b)+3+r>a+1+r,$$

we obtain

$$\begin{aligned}\varepsilon^a\partial_t w_\varepsilon-\Delta w_\varepsilon &\geq O(\varepsilon^{a+1+r})+O(\varepsilon^{2a-b+2+r})+O(\varepsilon^{2(a-b)+3+r})+\varepsilon^{a+1+\tau}\\ &=O(\varepsilon^{a+1+r})+\varepsilon^{a+1+\tau}\geq 0\end{aligned}$$

for ε small enough, provided

$$0<\tau<r. \tag{3.16}$$

Putting it all together, we first choose b satisfying

$$0<b<\min\{1,a\}.$$

Note that for $b<a$ and any integer k, we have that

$$(k+1)b-1<a+kb-1.$$

Therefore, since the quantity $\theta-(2-\gamma)[1-(k+1)b]$ is close to $(k+1)b-1$ when θ is close to 0 and γ is close to 1, we can choose θ sufficiently small and γ sufficiently close to 1 so that

$$0<\theta<a, \quad 1-\theta<\gamma<1,$$

and condition (3.12) is satisfied for $k=1,\dots,k_0$ with k_0 as in (3.11). Moreover, since the quantity $\theta+(2-\gamma)\frac{k+1}{2}a$ is close to $\frac{k+1}{2}a<(1+\frac{k}{2})a$, by eventually choosing θ smaller and γ closer to 1, condition (3.14) holds true for $k=0,\dots,k_1$ with k_1 as in (3.13). Finally, we choose $r>0$ satisfying (3.15) and $0<\tau<r$.

With this choice of coefficients, conditions (3.7), (3.10), (3.12), (3.14), and (3.16) are satisfied and the above computations show that w_ε is a solution to (3.6), as desired. This concludes the proof of the proposition. ∎

We next show that the function w_ε defined in (3.5) is above the initial condition (1.8) at initial time.

PROPOSITION 3.3. *There exist $\varepsilon_0,\delta_0>0$ such that for any $0<\varepsilon<\varepsilon_0$, if $(\bar z_1(0),\dots,\bar z_N(0))$ satisfies* (3.1) *with $0<\delta<\delta_0$, and b is as in Proposition* 3.2, *then the function w_ε defined as in* (3.5) *satisfies*

$$w_\varepsilon(x,y,0)\geq u^0_\varepsilon(x,y) \quad \textit{for all } (x,y)\in\overline{\mathbb{R}^2_+}, \tag{3.17}$$

with u^0_ε defined as in (1.8).

Proof. First note that, by the monotonicity of ϕ with respect to x, for every $i=1,\dots,N$,

$$\phi\left(\frac{x-\bar z_i(0)}{\varepsilon},\frac{y}{\varepsilon}\right)\geq\phi\left(\frac{x-z_i^0}{\varepsilon},\frac{y}{\varepsilon}\right). \tag{3.18}$$

By (2.9), there exists $K>0$ such that

$$\sup_{|x|>K}|\psi(x,y)|\sum_{i=1}^N|\bar c_i(0)|\leq\frac{\tilde\delta}{2}. \tag{3.19}$$

Moreover, by (2.12) with $R = 2\varepsilon^{-b}$, and recalling that $0 < b < a$, for ε small enough,

$$0 \le \varepsilon^{a+1} \sum_{i=1}^{N} |\bar{c}_i(0)| q\left(\frac{x - \bar{z}_i(0)}{\varepsilon}, \frac{y}{\varepsilon}\right) \le C\varepsilon^{a+1-b} |\ln \varepsilon| \le \frac{\varepsilon\tilde{\delta}}{2}. \tag{3.20}$$

Now, for fixed $(x,y) \in \overline{\mathbb{R}^2_+}$, we consider two cases.

Case 1: *There exists* $i_0 = 1, \dots, N$, *such that* $|x - \bar{z}_{i_0}(0)| \le \varepsilon K$. By the monotonicity of ϕ with respect to x,

$$\phi\left(\frac{x - \bar{z}_{i_0}(0)}{\varepsilon}, \frac{y}{\varepsilon}\right) \ge \phi\left(-K, \frac{y}{\varepsilon}\right),$$

while, for ε small enough,

$$\phi\left(\frac{x - z^0_{i_0}}{\varepsilon}, \frac{y}{\varepsilon}\right) = \phi\left(\frac{x - \bar{z}_{i_0}(0) - \delta}{\varepsilon}, \frac{y}{\varepsilon}\right) \le \phi\left(K - \frac{\delta}{\varepsilon}, \frac{y}{\varepsilon}\right) \le \phi\left(-\frac{\delta}{2\varepsilon}, \frac{y}{\varepsilon}\right).$$

Therefore, from (2.3),

$$\begin{aligned}
&\phi\left(\frac{x - \bar{z}_{i_0}(0)}{\varepsilon}, \frac{y}{\varepsilon}\right) - \phi\left(\frac{x - z^0_{i_0}}{\varepsilon}, \frac{y}{\varepsilon}\right) \\
&\ge \phi\left(-K, \frac{y}{\varepsilon}\right) - \phi\left(-\frac{\delta}{2\varepsilon}, \frac{y}{\varepsilon}\right) = \int_{-\frac{\delta}{2\varepsilon}}^{-K} \partial_x \phi\left(\tau, \frac{y}{\varepsilon}\right) \mathrm{d}\tau \\
&\ge C \int_{-\frac{\delta}{2\varepsilon}}^{-K} \frac{\frac{y}{\varepsilon} + 1}{\tau^2 + \left(\frac{y}{\varepsilon} + 1\right)^2} \mathrm{d}\tau \\
&\ge C\left(\frac{\delta}{2\varepsilon} - K\right) \frac{\frac{y}{\varepsilon} + 1}{\left(\frac{\delta}{\varepsilon}\right)^2 + \left(\frac{y}{\varepsilon} + 1\right)^2} \\
&\ge C\frac{\delta}{\varepsilon} \frac{\frac{y}{\varepsilon} + 1}{\left(\frac{\delta}{\varepsilon}\right)^2 + \left(\frac{y}{\varepsilon} + 1\right)^2} \\
&= C\delta \frac{y + \varepsilon}{\delta^2 + (y+\varepsilon)^2}.
\end{aligned}$$

By (2.8), for ε small enough, we infer that

$$\phi\left(\frac{x - \bar{z}_{i_0}(0)}{\varepsilon}, \frac{y}{\varepsilon}\right) - \phi\left(\frac{x - z^0_{i_0}}{\varepsilon}, \frac{y}{\varepsilon}\right) \ge \varepsilon \left|\bar{c}_{i_0}(0)\psi\left(\frac{x - \bar{z}_{i_0}(0)}{\varepsilon}, \frac{y}{\varepsilon}\right)\right|.$$

Next, since $|x - \bar{z}_{i_0}(0)| < \varepsilon K$, we have that $|x - \bar{z}_i(0)| > \varepsilon K$ for $i \ne i_0$, and by (3.19),

$$\sum_{i \ne i_0} \left|\bar{c}_i(0)\psi\left(\frac{x - \bar{z}_i(0)}{\varepsilon}, \frac{y}{\varepsilon}\right)\right| \le \frac{\tilde{\delta}}{2}.$$

Combining the two last estimates with (3.18) and (3.20) yields

$$\begin{aligned}
w_\varepsilon(x,y,0) = &\sum_{i=1}^{N} \left[\phi\left(\frac{x - \bar{z}_i(0)}{\varepsilon}, \frac{y}{\varepsilon}\right) - \varepsilon\bar{c}_i(0)\psi\left(\frac{x - \bar{z}_i(0)}{\varepsilon}, \frac{y}{\varepsilon}\right)\right] + \varepsilon\tilde{\delta} \\
&+ \varepsilon^{a+1} \sum_{i=1}^{N} \bar{c}_i(0) q\left(\frac{x - \bar{z}_i(0)}{\varepsilon}, \frac{y}{\varepsilon}\right) + \varepsilon^{\theta}(y+\varepsilon)^{\gamma}
\end{aligned}$$

$$
\begin{aligned}
&\geq \left[\phi\left(\frac{x-\bar z_{i_0}(0)}{\varepsilon},\frac{y}{\varepsilon}\right)-\varepsilon\bar c_{i_0}(0)\psi\left(\frac{x-\bar z_{i_0}(0)}{\varepsilon},\frac{y}{\varepsilon}\right)\right]+\sum_{i\neq i_0}\phi\left(\frac{x-\bar z_i(0)}{\varepsilon},\frac{y}{\varepsilon}\right)\\
&\quad-\varepsilon\sum_{i\neq i_0}\bar c_i(0)\psi\left(\frac{x-\bar z_i(0)}{\varepsilon},\frac{y}{\varepsilon}\right)+\varepsilon^{a+1}\sum_{i=1}^N\bar c_i(0)q\left(\frac{x-\bar z_i(0)}{\varepsilon},\frac{y}{\varepsilon}\right)+\varepsilon\tilde\delta\\
&\geq\phi\left(\frac{x-z_{i_0}^0}{\varepsilon},\frac{y}{\varepsilon}\right)+\sum_{i\neq i_0}\phi\left(\frac{x-z_i^0}{\varepsilon},\frac{y}{\varepsilon}\right)-\frac{\varepsilon\tilde\delta}{2}-\frac{\varepsilon\tilde\delta}{2}+\varepsilon\tilde\delta\\
&=\sum_{i=1}^N\phi\left(\frac{x-z_i^0}{\varepsilon},\frac{y}{\varepsilon}\right)=u_\varepsilon^0(x,y),
\end{aligned}
$$

as desired.

Case 2: $|x-\bar z_{i_0}(0)|>\varepsilon K$ *for all* $i=1,\dots,N$. By (3.19),

$$\sum_{i=1}^N\left|\bar c_i(0)\psi\left(\frac{x-\bar z_i(0)}{\varepsilon},\frac{y}{\varepsilon}\right)\right|\leq\frac{\tilde\delta}{2},$$

which together with (3.18) and (3.20) implies

$$
\begin{aligned}
w_\varepsilon(x,y,0)&\geq\sum_{i=1}^N\phi\left(\frac{x-\bar z_i(0)}{\varepsilon},\frac{y}{\varepsilon}\right)-\varepsilon\sum_{i=1}^N\bar c_i(0)\psi\left(\frac{x-\bar z_i(0)}{\varepsilon},\frac{y}{\varepsilon}\right)\\
&\quad+\varepsilon^{a+1}\sum_{i=1}^N\bar c_i(0)q\left(\frac{x-\bar z_i(0)}{\varepsilon},\frac{y}{\varepsilon}\right)+\varepsilon\tilde\delta\\
&\geq\sum_{i=1}^N\phi\left(\frac{x-z_i^0}{\varepsilon},\frac{y}{\varepsilon}\right)=u_\varepsilon^0(x,y).
\end{aligned}
$$

From Cases 1 and 2, we infer that (3.17) holds for every $(x,y)\in\overline{\mathbb{R}^2_+}$. This completes the proof of the proposition. ∎

Subsolutions to (1.1) with initial datum (1.8) are constructed in a manner similar to that of supersolutions. Consider the perturbed system, for $i=1,\dots,N$, $\delta>0$,

$$
\begin{cases}
\dfrac{d\underline{z}_i}{dt}=\dfrac{c_0}{\pi}\left(\displaystyle\sum_{j\neq i}\frac{1}{\underline{z}_i-\underline{z}_j}+\delta\right), & t>0,\\
\underline{z}_i(0)=z_i^0+\delta,
\end{cases}
\tag{3.21}
$$

and let

$$\underline{c}_i(t):=\dot{\underline{z}}_i(t),$$

and $\tilde\delta$ let be defined as in (3.2). Then one can prove that the function

$$
\begin{aligned}
h_\varepsilon(x,y,t):=&\sum_{i=1}^N\left[\phi\left(\frac{x-\underline{z}_i(t)}{\varepsilon},\frac{y}{\varepsilon}\right)-\varepsilon\underline{c}_i(t)\psi\left(\frac{x-\underline{z}_i(t)}{\varepsilon},\frac{y}{\varepsilon}\right)\right]-\varepsilon\tilde\delta\\
&+\varepsilon^{a+1}\sum_{i=1}^N\underline{c}_i(t)q\left(\frac{x-\underline{z}_i(t)}{\varepsilon},\frac{y}{\varepsilon}\right)\\
&-\varepsilon^\theta(y+\varepsilon)^\gamma-\varepsilon^{1+\tau}t
\end{aligned}
\tag{3.22}
$$

is a subsolution to (1.1) with initial datum (1.8). More precisely, we have the following.

PROPOSITION 3.4. *Given $T>0$ and $a>0$, there exist ε_0, δ_0, $\tau,\theta>0$ and $0<b,\gamma<1$ such that, for any $0<\varepsilon<\varepsilon_0$, if $(\underline{z}_1(t),\dots,\underline{z}_N(t))$ is the solution of* (3.21) *with $0<\delta<\delta_0$, then the function h_ε defined in* (3.22) *solves*

$$\begin{cases}\varepsilon^a\partial_t h_\varepsilon-\Delta h_\varepsilon\le 0, & y>0,\, t\in(0,T),\\ \varepsilon\partial_t h_\varepsilon-\partial_y h_\varepsilon+\frac{1}{\varepsilon}W'(h_\varepsilon)\le 0, & y=0, t\in(0,T),\end{cases}$$

and

$$h_\varepsilon(x,y,0)\le u^0_\varepsilon(x,y)\quad \textit{for all } (x,y)\in\overline{\mathbb{R}^2_+}.$$

4. Proof of Theorem 1.1. In this section, we complete the proof of Theorem 1.1 using the constructed super/subsolutions, the comparison principle, and the decay estimates established in previous sections.

Proof of Theorem 1.1. Let w_ε and h_ε be the functions defined in (3.5) and (3.22). Given any $T>0$, by Propositions 3.2, 3.3, and 3.4 there exist δ_0, $\varepsilon_0>0$ and coefficients $\theta,\tau>0$, $0<b,\gamma<1$ such that for $0<\varepsilon<\varepsilon_0$ and $0<\delta<\delta_0$, w_ε and h_ε are, respectively, the super- and subsolution to (1.1) with initial datum (1.8) in $\mathbb{R}^2_+\times[0,T]$. Since w_ε and h_ε are strictly sublinear in y, we can apply the comparison principle to conclude that

$$h_\varepsilon(x,y,t)\le u_\varepsilon(x,y,t)\le w_\varepsilon(x,y,t)\quad \text{for all } (x,y,t)\in\overline{\mathbb{R}^2_+}\times[0,T]. \tag{4.1}$$

Note that from (2.12) with $R=2\varepsilon^{-b}$, $0<b<1$,

$$\varepsilon^{a+1}\sum_{i=1}^N\bar c_i(t)q\left(\frac{x-\bar z_i(t)}{\varepsilon},\frac{y}{\varepsilon}\right),\, \varepsilon^{a+1}\sum_{i=1}^N\bar c_i(t)q\left(\frac{x-\underline{z}_i(t)}{\varepsilon},\frac{y}{\varepsilon}\right)\to 0\quad \text{as } \varepsilon\to 0^+.$$

Let $(x,y,t)\in\overline{\mathbb{R}^2_+}\times[0,\infty)$. Then, from (4.1),

$$\limsup_{\varepsilon\to0^+}u_\varepsilon(x,y,t)\le\limsup_{\varepsilon\to0^+}\sum_{i=1}^N\phi\left(\frac{x-\bar z_i(t)}{\varepsilon},\frac{y}{\varepsilon}\right) \tag{4.2}$$

and

$$\liminf_{\varepsilon\to0^+}u_\varepsilon(x,y,t)\ge\liminf_{\varepsilon\to0^+}\sum_{i=1}^N\phi\left(\frac{x-\underline{z}_i(t)}{\varepsilon},\frac{y}{\varepsilon}\right), \tag{4.3}$$

since the other terms in w_ε and h_ε vanish when ε goes to 0. From (2.5), when $y>0$,

$$\limsup_{\varepsilon\to0^+}u_\varepsilon(x,y,t)\le\frac{1}{\pi}\sum_{i=1}^N\left(\frac{\pi}{2}+\arctan\left(\frac{x-\bar z_i(t)}{y}\right)\right)$$

and

$$\liminf_{\varepsilon\to0^+}u_\varepsilon(x,y,t)\ge\frac{1}{\pi}\sum_{i=1}^N\left(\frac{\pi}{2}+\arctan\left(\frac{x-\underline{z}_i(t)}{y}\right)\right).$$

Sending $\delta\to0$ yields statement (ii) of Theorem 1.1.

Statement (i) follows from (4.2), (4.3), and the following lemma. This completes the proof of Theorem 1.1. ∎

LEMMA 4.1. *Let* v_0 *be defined as in* (1.10), *let* ϕ *be the stationary layer solution solving* (1.7), *and let* $\bar{z}_i$, $\underline{z}_i$ *solve ODEs* (3.1), (3.21), *respectively. Then we have*

$$\limsup_{\delta\to 0^+}\ \limsup_{\substack{(x',y',t')\to(x,0,t)\\ \varepsilon\to 0^+}} \sum_{i=1}^N \phi\left(\frac{x'-\bar{z}_i(t')}{\varepsilon},\frac{y'}{\varepsilon}\right) \le (v_0)^*(x,t) \tag{4.4}$$

and

$$\liminf_{\delta\to 0^+}\ \liminf_{\substack{(x',y',t')\to(x,0,t)\\ \varepsilon\to 0^+}} \sum_{i=1}^N \phi\left(\frac{x'-\underline{z}_i(t')}{\varepsilon},\frac{y'}{\varepsilon}\right) \ge (v_0)_*(x,t). \tag{4.5}$$

Proof. Let us prove (4.4). The proof of (4.5) follows with a similar argument. Let H^* be defined by $H^*(s)=H(s)$ if $s\neq 0$ and $H^*(0)=1$. It is easy to prove that

$$(v_0)^*(x,t)=\sum_{i=1}^N H^*(x-z_i(t)).$$

Fix (x,t) and consider two cases.

Case 1: *There exists* i_0 *such that* $x=z_{i_0}(t)$. Let (x_n,y_n,t_n) be a sequence converging to $(x,0,t)$. By Lemma 2.1, for δ small enough, we have that

$$x-\bar{z}_i(t)>0 \text{ for } i=1,\dots,i_0-1 \quad\text{and}\quad x-\bar{z}_i(t)<0 \text{ for } i=i_0+1,\dots,N,$$

and for ε small enough and n large enough,

$$x_n-\bar{z}_i(t_n)\ge\varepsilon^{\frac14} \text{ for } i=1,\dots,i_0-1 \quad\text{and}\quad x_n-\bar{z}_i(t_n)\le-\varepsilon^{\frac14} \text{ for } i=i_0+1,\dots,N.$$

Assume that $y_n=0$ for all n. Then, by the monotonicity of ϕ and its behavior at infinity, we get

$$\lim_{\substack{n\to\infty\\ \varepsilon\to 0^+}} \phi\left(\frac{x_n-\bar{z}_i(t_n)}{\varepsilon},0\right)=\begin{cases}1 & \text{if } i=1,\dots,i_0-1\\ 0 & \text{if } i=i_0+1,\dots,N\end{cases}=H^*(x-\bar{z}_i(t)).$$

By (2.5), the same limit holds true when $y_n>0$ and $y_n\to 0$.

When $i=i_0$, since $\phi<1$ we simply have

$$\limsup_{\substack{n\to\infty\\ \varepsilon\to 0^+}} \phi\left(\frac{x_n-\bar{z}_{i_0}(t_n)}{\varepsilon},\frac{y_n}{\varepsilon}\right)\le 1=H^*(x-z_{i_0}(t)).$$

Since the limits above are computed along any arbitrary sequence (x_n,y_n,t_n) converging to $(x,0,t)$, we conclude that

$$\begin{aligned}&\limsup_{\delta\to 0^+}\ \limsup_{\substack{(x',y',t')\to(x,0,t)\\ \varepsilon\to 0^+}} \sum_{i=1}^N \phi\left(\frac{x'-\bar{z}_i(t')}{\varepsilon},\frac{y'}{\varepsilon}\right)\\ &\quad\le \limsup_{\delta\to 0^+}\sum_{i=1}^N\ \limsup_{\substack{(x',y',t')\to(x,0,t)\\ \varepsilon\to 0^+}} \phi\left(\frac{x'-\bar{z}_i(t')}{\varepsilon},\frac{y'}{\varepsilon}\right)\end{aligned}$$

$$\leq \limsup_{\delta\to 0^+} \sum_{i\neq i_0}^{N} H^*(x-\bar{z}_i(t)) + H^*(x - z_{i_0}(t))$$
$$= (v_0)^*(x,t).$$

This proves (4.4) in Case 1.

Case 2: $x \neq z_i(t)$ *for all* $i = 1,\ldots,N$. Arguing as in Case 1, we obtain that, for all $i = 1,\ldots,N$,

$$\limsup_{\substack{(x',y',t')\to(x,0,t)\\ \varepsilon\to 0^+}} \phi\left(\frac{x'-\bar{z}_i(t')}{\varepsilon}, \frac{y'}{\varepsilon}\right) = H^*(x-\bar{z}_i(t)),$$

from which (4.4) follows. ∎

REFERENCES

[Bow09] A. F. Bower, *Applied Mechanics of Solids*, CRC Press, 2009.

[Che04] X. Chen, *Generation, propagation, and annihilation of metastable patterns*, J. Differential Equations, 206 (2004), pp. 399–437, https://doi.org/10.1016/j.jde.2004.05.017.

[CP89] J. Carr and R. L. Pego, *Metastable patterns in solutions of* $u_t = \epsilon^2 u_{xx} - f(u)$, Comm. Pure Appl. Math., 42 (1989), pp. 523–576, https://doi.org/10.1002/cpa.3160420502.

[CS07] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations, 32 (2007), pp. 1245–1260, https://doi.org/10.1080/03605300600987306.

[CSM05] X. Cabré and J. Solà-Morales, *Layer solutions in a half-space for boundary reactions*, Commun. Pure Appl. Math., 58 (2005), pp. 1678–1732, https://doi.org/10.1002/cpa.20093.

[DFV14] S. Dipierro, A. Figalli, and E. Valdinoci, *Strongly nonlocal dislocation dynamics in crystals*, Comm. Partial Differential Equations, 39 (2014), pp. 2351–2387, https://doi.org/10.1080/03605302.2014.914536.

[DPV15] S. Dipierro, G. Palatucci, and E. Valdinoci, *Dislocation dynamics in crystals: A macroscopic theory in a fractional Laplace setting*, Comm. Math. Phys., 333 (2015), pp. 1061–1105, https://doi.org/10.1007/s00220-014-2118-6.

[FIM09] N. Forcadel, C. Imbert, and R. Monneau, *Homogenization of some particle systems with two-body interactions and of the dislocation dynamics*, Discrete Contin. Dyn. Syst., 23 (2009), pp. 785–826, https://doi.org/10.3934/dcds.2009.23.785.

[GM12] M. González and R. Monneau, *Slow motion of particle systems as a limit of a reaction-diffusion equation with half-Laplacian in dimension one*, Discrete Contin. Dyn. Syst., 32 (2012), pp. 1255–1286, https://doi.org/10.3934/dcds.2012.32.1255.

[GR23] Y. Gao and J.-M. Roquejoffre, *Asymptotic stability for diffusion with dynamic boundary reaction from Ginzburg–Landau energy*, SIAM J. Math. Anal., 55 (2023), pp. 1246–1263, https://doi.org/10.1137/22M1469791.

[MP12] R. Monneau and S. Patrizi, *Derivation of Orowan's law from the Peierls-Nabarro model*, Comm. Partial Differential Equations, 37 (2012), pp. 1887–1911, https://doi.org/10.1080/03605302.2012.683504.

[Nab47] F. R. N. Nabarro, *Dislocations in a simple cubic lattice*, Proc. Phys. Soc., 59 (1947), pp. 256–272, https://doi.org/10.1088/0959-5309/59/2/309.

[Pei40] R. Peierls, *The size of a dislocation*, Proc. Phys. Soc., 52 (1940), pp. 34–37, https://doi.org/10.1088/0959-5309/52/1/305.

[PS21] S. Patrizi and T. Sangsawang, *From the Peierls-Nabarro model to the equation of motion of the dislocation continuum*, Nonlinear Anal., 202 (2021), 112096, https://doi.org/10.1016/j.na.2020.112096.

[PS23] S. Patrizi and T. Sangsawang, *Derivation of the* 1*-D Groma-Balogh equations from the Peierls-Nabarro model*, Calc. Var. Partial Differential Equations, 62 (2023), 242, https://doi.org/10.1007/s00526-023-02575-7.

[PV15] S. Patrizi and E. Valdinoci, *Crystal dislocations with different orientations and collisions*, Arch. Ration. Mech. Anal., 217 (2015), pp. 231–261, https://doi.org/10.1007/s00205-014-0832-z.

[PV16] S. Patrizi and E. Valdinoci, *Relaxation times for atom dislocations in crystals*, Calc. Var. Partial Differential Equations, 55 (2016), pp. 1–44, https://doi.org/10.1007/s00526-016-1000-0.

[PV17] S. Patrizi and E. Valdinoci, *Long-time behavior for crystal dislocation dynamics*, Math. Models Methods Appl. Sci., 27 (2017), pp. 2185–2228, https://doi.org/10.1142/S0218202517500427.

[SZdQ+22] J. Song, Q. Zhang, F. de Quesada, M. H. Rizvi, J. B. Tracy, J. Ilavsky, S. Narayanan, E. D. Gado, and R. L. Leheny, N. Holten-Andersen, and G. H. McKinley, *Microscopic dynamics underlying the stress relaxation of arrested soft materials*, Proc. Natl. Acad. Sci. USA, 119 (2022), e2201566119, https://doi.org/10.1073/pnas.2201566119.

[VMPP22] P. Van Meurs, M. A. Peletier, and N. Požár, *Discrete-to-continuum convergence of charged particles in* 1*D with annihilation*, Arch. Ration. Mech. Anal., 246 (2022), pp. 241–297, https://doi.org/10.1007/s00205-022-01812-1.